MAT344 Week 10

2019/Nov/25

1 This week

This week, we are talking about

1. Exponential generating functions
2. Recurrences

2 Recap

Last week we talked about

1. Generating functions

3 Exponential generating functions

We declared that the function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

is the generating function for the sequence a_n (that probably represents the answer to a counting problem). These sort of generating functions are based on the function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We could have chosen another series to base the encoding of our counting problems on, for example, the Taylor series of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Definition 3.1 (Definition 9.6. in [Mor17])**.** *The **exponential generating function** of a sequence $a_0, a_1, \dots$ is*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

So, the coefficients of an exponential generating function and an ordinary generating function differ by a factor of $n!$. We will also use exponential generating functions to encode answers to counting problems, and we'll see that they are useful when we consider objects that are distinguishable. Ordinary generating functions (those based on $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$) are useful for counting indistinguishable objects.

Example 3.2. *The exponential generating function for the number of binary strings of length n is*

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}.$$

compare this with the ordinary generating function for the same sequence

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

The difference between the two functions e^{2x} and $\frac{1}{1-2x}$ is that they have different algebraic properties, and this makes one or the other more convenient in certain situations. For example, the way exponential generating functions multiply is more convenient to use when the order matters. Why is this? Consider how exponential generating functions multiply. If $a(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$ and $b(x) = \sum_{l=0}^{\infty} b_l \frac{x^l}{l!}$, then

$$a(x)b(x) = \sum_{n=0}^{\infty} \sum_{k+l=n} a_k \frac{x^k}{k!} b_l \frac{x^l}{l!} \qquad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \frac{n!}{k!(n-k)!} \frac{x^n}{n!} \qquad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \binom{n}{k} \frac{x^n}{n!} \tag{1}$$

and we see the coefficient $\frac{n!}{k!l!} = \frac{n!}{k!(n-k)!}$ appear, which corresponds to the number of binary strings of length n with exactly k 0s.

For example, note that in the example above, we could represent a 0-string of length n (or a 1-string of length n) by the exponential generating function e^x or the ordinary generating function $\frac{1}{1-x}$. If we want to make a binary string of length n, we can just multiply the two exponential generating functions together

$$e^x e^x = e^{2x}$$

and get the correct answer. If we multiply the two ordinary generating functions together

$$\frac{1}{1-x}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

then we get the number of ways of picking a total n 0s and 1s, i.e. a $\{0,1\}$-combination with repetition. We could also interpret this as $\{0,1\}$-strings when anagrams are considered the same, i.e. $\{0,1\}$-strings of length n up to permutation.

Example 3.3 (Example 8.16 in [KT17])**.** *Find the number of ternary strings in which the number of 0s is even.*

Solution: The exponential generating function for the 1s and 2s in the sequence is just e^x. To get an even number of 0s, we need the function

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

(Note that this is not the function e^{2x}). How can we represent this as a function? Note that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

and therefore

$$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \ldots$$

so the exponential generating function for the 0s is $\frac{e^x+e^{-x}}{2}$.

Then we multiply the exponential generating functions for the letters together to get

$$\frac{e^x + e^{-x}}{2} e^x e^x = \frac{e^{3x} + e^x}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right).$$

so the number of ternary strings with an even number of 0s is

$$\frac{3^n + 1}{2}.$$

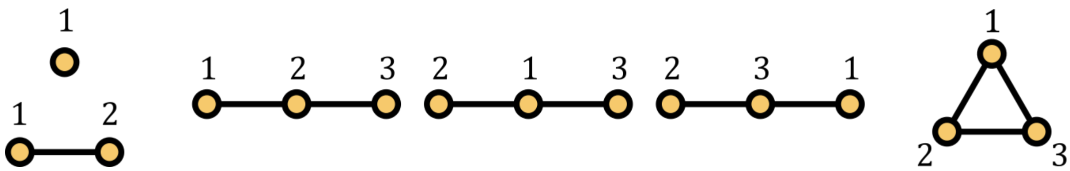

Figure 1: The six labeled, connected graphs with at most 3 vertices

4 Connected labeled graphs on n vertices (Chapter 3.1 in [Wil90])

Figure 1 shows connected labeled graphs with at most 3 vertices. There is just one on one vertex, one on two vertices, and four on three vertices. How can we count how many connected labeled graphs are there on n vertices?

If we consider graphs that are not necessarily connected, then there are at most $\binom{n}{2}$ edges, each of which may or may not be part of the labeled graph. So we established the following:

Proposition 4.1. *There are $2^{\binom{n}{2}}$ labeled graphs on n vertices.*

So the exponential generating function for the number of labeled graphs is

$$L(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}.$$

Connected labeled graphs are the bulding blocks of labeled graphs, so there *should be* a formula relating the number of connected labeled graphs to the number of labeled graphs.

Let c_n be the number of connected labeled graphs on n vertices and form the exponential generating function

$$C(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

How exactly is a labeled graph on n vertices built out of its connected components?

In a situation where there is a set of labeled "building blocks" for a kind of object, there is a formula relating the exponential generating functions counting the building blocks and counting the objects.

Theorem 4.2 (Exponential formula, Corollary 3.4.1 in [Wil90])**.** *Let o_n be the number of some labeled objects (depending on some integer n), which may consist of several blocks. Let k_n be the number of possible blocks of a fixed size. Let $O(x) = \sum_{n=0}^{\infty} o_n \frac{x^n}{n!}$ and $K(x) = \sum_{n=0}^{\infty} k_n \frac{x^n}{n!}$ be the exponential generating functions of the objects and blocks, respectively. The exponential generating functions $O(x)$ and $K(x)$ satisfy*

$$O(x) = e^{K(x)}.$$

Note that $L(x)$ does not converge anywhere, but this should not deter us from working with its formal power series.

$$\begin{aligned} L(x) &= e^{C(x)} && \\ \ln L(x) &= C(x) && \text{taking log of both sides} \\ \frac{L'(x)}{L(x)} &= C'(x) && \text{differentiating both sides} \\ L'(x) &= C'(x)L(x) && \text{clearing denominators} \\ xL'(x) &= xC'(x)L(x) && \text{multiplying by } x \end{aligned}$$

Now let's compute these two series. On the left hand-side, we have

$$xL'(x) = \sum_{n=1}^{\infty} n 2^{\binom{n}{2}} \frac{x^n}{n!}$$

(note that the constant term disappears). For the right-hand side, we compute

$$xC'(x) = \sum_{n=1}^{\infty} nc_n \frac{x^n}{n!}, \qquad L(x)\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}.$$

Using (1), we see that

$$xC'(x)L(x) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \binom{n}{k} kc_k 2^{\binom{n-k}{2}} \frac{x^n}{n!}.$$

Comparing coefficients of $\frac{x^n}{n!}$ yields

$$n2^{\binom{n}{2}} = \sum_{k=1}^{n} \binom{n}{k} kc_k 2^{\binom{n-k}{2}}. \tag{2}$$

We can use this to compute the first few values of c_n, and we find $1, 1, 4, 38, 728, 26704\ldots$ (no closed formula is known).

Exercise 4.3. *Find a combinatorial proof for equation (2).* ***Hint:*** *The left hand side looks like we are counting connected labeled graphs with a distinguished vertex.*

5 Set partitions

In Tutorial 6, you studied **partitions of** $[n]$. Recall that a partition of $[n]$ into k parts is a collection of disjoint nonempty subsets whose union is $[n]$. For example, $\{\{1,3\},\{2\},\{4\}\}$ is a partition of $[4]$ into 3 parts. We denoted the number of partitions of $[n]$ into k parts as $\left\{ {n \atop k} \right\}$ and deduced some recursive formulas to compute them.

Definition 5.1. *The* ***Bell number*** B_n *is the total number of partitions of* $[n]$*, i.e.*

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}.$$

We can use Theorem 4.2 to find an exponential generating function for the Bell numbers. The objects are the partitions, and the building blocks are the disjoint subsets. Let $B(x) = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}$ be the exponential generating function for the Bell numbers.

In this case, the building blocks are just sets so they are easy to count, there is exactly one of them of each size n for $n \geq 1$. So the exponential generating function for them is

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1.$$

So by Theorem 4.2, we have

$$B(x) = e^{e^x-1}.$$

Note that this might not be very useful, but a few quick sage commands

```
var('x')
f=exp(exp(x)-1)
f.series(x,8)
```

will produce the output

```
1 + 1*x + 1*x^2 + 5/6*x^3 + 5/8*x^4 + 13/30*x^5 + 203/720*x^6 + 877/5040*x^7 + Order(x^8)
```

From which we can easily read off the first few Bell numbers

$$1, 1, 2, 5, 15, 52, 203, 877, \ldots$$

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 2

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 1

[Wil90] Herbert S. Wilf. *Generatingfunctionology*. Academic Press, 1990. Available at https://www.math.upenn.edu/~wilf/DownldGF.html. 3