MAT344 Week 11

2019/Nov/25

1 This week

This week, we are talking about

1. Recurrence equations

2 Recap

Last week we talked about

1. Exponential generating functions

3 Recurrence Equations (Ch. 9 in [KT17])

We have already seen recurrences earlier in the semester where we used them to prove that certain objects are counted by the same formula. For example, we showed that Catalan numbers C_n count triangulations of a convex polygon by showing that they satisfy the same recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

and have the same initial conditions. Similarly, we argued that domino tilings of a $2 \times n$ rectangle are counted by Fibonacci numbers.

Last week, using generating functions, we were able to "solve" the recurrence equation

$$a_n = 3a_{n-1} - 1$$

and $a_0 = 2$. What do we mean by solving a recurrence equation? We were able to establish the formula

$$a_n = \frac{1}{2} + \frac{3}{2} 3^n.$$

A formula for a sequence is always better than a recursion. Just think about the number of operations required to compute a_n using the recursive definition versus plugging in into the formula.

4 Linear Recurrence Equations (Ch. 9.2 in [KT17])

Consider the Fibonacci recurrence

$$F_n - F_{n-1} - F_{n-2} = 0.$$

this is a **linear recurrence**, because we can compute the n-th term as a linear combination of the previous terms.

Definition 4.1. *A **linear recurrence** is a recurrence equation of the form*

$$c_0 a_{n+k} + c_1 a_{n+k-1} + \ldots + c_k a_n = g(n).$$

Compare this to the Catalan recurrence, which is not linear. The RHS of the Fibonacci recurrence is zero, and we will refer to this as a **homogeneous** linear recurrence.

5 Advancement Operators (Ch. 9.3 in [KT17])

The theory of linear recurrence equations is very similar to that of linear differential equations.

Example 5.1 (Example 9.4. in [KT17]). *Solve the differential equation*

$$\frac{d}{dx}f(x) = 3f(x)$$

with the initial condition $f(0) = 2$.

If you have seen differential equations, you know that the solution is $f(x) = 2e^{3x}$.

For differential equations, we apply the operator $\frac{d}{dx}$ (or its powers) to a (differentiable) function, and look for a solution.

For recurrence equations, we replace differentiable functions by sequences of numbers (or, functions $f : \mathbb{Z} \to \mathbb{R}$), and the operator $\frac{d}{dx}$ by the **advancement operator** A defined by

$$Af(n) = f(n+1).$$

For example, we could represent the Fibonacci sequence by the function

$$F(n) = F_n,$$

and then the recurrence, rewritten in terms of the advancement operator is

$$(A^2 - A - 1)f(n) = A^2 f(n) - Af(n) - f(n) = 0.$$

Before we solve this equation, let's take a look at an easier one.

Example 5.2 (Example 9.6 in [KT17]). *Suppose that the sequence $\{s_n | n \geq 0\}$ satisfies $s_0 = 3$ and $s_{n+1} = 2s_n$ for $n \geq 1$. Find an explicit formula for s_n.*

Solution: After some thought, we can guess that the solution is $s_n = 3 \cdot 2^n$, but let us rewrite it in terms of the advancement operator. We have

$$\begin{aligned} As(n) &= 2s(n) \\ As(n) - 2s(n) &= 0 \\ (A-2)s(n) &= 0 \end{aligned}$$

Notice that the advancement polynomial $(A - 2)$ has a root exactly at 2, and the solution is $s_n = 3 \cdot 2^n$.

Example 5.3 (Example 9.7 in [KT17]). *Find all solutions to the advancement operator equation*

$$(A^2 + A - 6)f(n) = 0.$$

Solution: We factor the polynomial $A^2 + A - 6 = (A + 3)(A - 2)$. If we write the equation now, we see that

$$(A+3)(A-2)f(n) = 0. \tag{1}$$

Note that any solution to $(A - 2)f(n) = 0$ or $(A + 3)f(n) = 0$ is a solution to equation (1).

For example, if $f(n) = c2^n$ for some constant c (as in the previous example), then $f(n)$ is still a solution. Also, by a similar logic, any function of the form $f(n) = c(-3)^n$ is also a solution. We will try to find all the solutions. Let $f(n) = c_1 2^n + c_2(-3)^n$, and apply $(A + 3)(A - 2)$. We have

$$\begin{aligned} (A+3)(A-2)f(n) &= (A+3)(c_1 2^{n+1} + c_2(-3)^{n+1} - 2(c_1 2^n + c_2(-3)^n)) \\ &= (A+3)(-5c_2(-3)^n) \\ &= 0. \end{aligned}$$

6 Linear Homogeneous Recurrence Equations (Ch. 9.2 in [KT17])

Now we consider a more general case.

Proposition 6.1. *If we have a polynomial of the form $p(A) = (A - k_1)(A - k_2)\dots(A - k_l)$ in the advancement operator A, and our recurrence equation is*

$$p(A)f(n) = 0,$$

then all the solutions to the recurrence are of the form

$$f(n) = c_1 k_1^n + c_2 k_2^n + \dots + c_l k_l^n.$$

Exercise 6.2. *Solve the linear homogeneous recurrence*

$$f(n) = f(n-1) + f(n-2).$$

with $f(0) = 0, f(1) = 1$.

Solution: We rewrite the equation in terms of the advancement operator as

$$(A^2 - A - 1)f(n) = 0.$$

the polynomial factors into linear factors

$$(A^2 - A - 1) = \left(A - \frac{\sqrt{5}+1}{2}\right)\left(A + \frac{\sqrt{5}-1}{2}\right).$$

by Proposition 6.1,

$$f(n) = c_1 \left(\frac{\sqrt{5}+1}{2}\right)^n + c_2 \left(-\frac{\sqrt{5}-1}{2}\right)^n$$

To find what c_1 and c_2 are, we have to use the initial conditions $f(0) = 0, f(1) = 1$. Plugging in $n = 0$ into our formula we get

$$c_1 + c_2 = 0,$$

so, $c_2 = -c_1$. Using this and plugging in $n = 1$ we get

$$\begin{aligned}
c_1 \left(\frac{\sqrt{5}+1}{2}\right) + c_2 \left(-\frac{\sqrt{5}-1}{2}\right) &= 1 \\
c_1 \left(\frac{\sqrt{5}+1}{2}\right) - c_1 \left(-\frac{\sqrt{5}-1}{2}\right) &= 1 \\
c_1 \left(\frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2}\right) &= 1 \\
c_1 \left(\sqrt{5}\right) &= 1 \\
c_1 &= \frac{1}{\sqrt{5}}
\end{aligned}$$

Therefore

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

The formula looks strange, it not only involves fractions, but also square roots. The recurrence makes it clear that $f(n)$ is an integer, but this is far from obvious from the formula. However, if we use the binomial theorem, we see that

$$\begin{aligned}
f(n) &= \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\left(\binom{n}{k}(\sqrt{5})^k\right) - \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\left(\binom{n}{k}(-1)^k(\sqrt{5})^k\right) \\
&= \frac{1}{2^n\sqrt{5}}\sum_{k=0}^{n}\left(\binom{n}{k}(\sqrt{5})^k(1-(-1)^k)\right) \\
&= \frac{1}{2^n\sqrt{5}}\sum_{k=0,\text{ odd}}^{n}\left(2\binom{n}{k}(\sqrt{5})^k\right) \\
&= \frac{1}{2^n}\sum_{k\text{ odd, }0\le k\le n}\left(2\binom{n}{k}(\sqrt{5})^{k-1}\right)
\end{aligned}$$

So the odd powers of $\sqrt{5}$ all cancel.

It is also worthwhile to point out that since

$$\left|\frac{1-\sqrt{5}}{2}\right| < 1,$$

we have

$$\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0.$$

So eventually

$$f(n) \sim \left(\frac{1+\sqrt{5}}{2}\right)^n$$

7 Multiple roots

Example 7.1. *Find all solutions of the advancement operator equation*

$$(A-2)^2 f = 0.$$

Solution: We see immediately that $f(n) = c_1 2^n$ is a solution, since it satisfies $(A-2)f = 0$. But there should probably be more solutions than just this. If you are familiar with differential equations, you can probably guess that we should try $f(n) = c_2 n 2^n$. Indeed,

$$\begin{aligned}
(A-2)^2 c_2 n 2^n &= (A-2)\left(c_2(n+1)2^{n+1} - 2c_2 n 2^n\right) \\
&= (A-2)\left(c_2 2^{n+1}\right) \\
&= 0.
\end{aligned}$$

Also notice that $f(n) = c_2 n 2^n$ does not satisfy $(A-2)f(n) = 0$, so this is a different solution from $c_1 2^n$. In this case, all solutions are of the form

$$f(n) = c_1 2^n + c_2 n 2^n.$$

Exercise 7.2 (Example 9.11. in [KT17])**.** *Find all the solutions of the advancement operator equation*

$$(A+5)(A-1)^3 f = 0.$$

Hint: try cn^2.

8 Nonhomogeneous equations (Ch. 9.4.2 in [KT17])

Example 8.1 (Example 9.12. in [KT17])**.** *Find all solutions of the advancement operator equation*

$$(A+2)(A-6)f = 3^n. \tag{2}$$

Solution: We already know how to solve the homogeneous equation

$$(A+2)(A-6)f = 0,$$

which has solutions $c_1(-2)^n + c_2 6^n$.

Therefore if we can just find one solution (a **particular solution**) to the nonhomogeneous equation (2), then we can add any solution to the particular solution to get another particular solution. It turns out that all solutions are of this form.

How do we guess what a particular solution should be? This is not easy in general, but a reasonable guess would be $c3^n$. If $f_0(n) = c3^n$, then

$$\begin{aligned}(A+2)(A-6)f_0(n) &= (A+2)(c3^{n+1} - 6c3^n) \\ &= (A+2)(-c3^{n+1}) \\ &= -c3^{n+2} - 2c3^{n+1} \\ &= -5c(3^{n+1}).\end{aligned}$$

If f_0 is a solution to equation (2), then we want

$$-5c(3^{n+1}) = 3^n,$$

i.e. $c = -\frac{1}{15}$. Then the general solution is

$$f(n) = c_1(-2)^n + c_2 6^n - \frac{1}{15}3^n.$$

for constants c_1, c_2.

9 Lines and regions in the plane

Example 9.1. *Draw n lines in the plane in a way that any two lines intersect at exactly one point and no point in the plane belongs to two lines. Let r_n be the number of regions that the lines divide the plane into. We have $r_0 = 1, r_1 = 2, r_2 = 4$. From Figure 1, we see that $r_4 = 11$. Find a formula for r_n.*

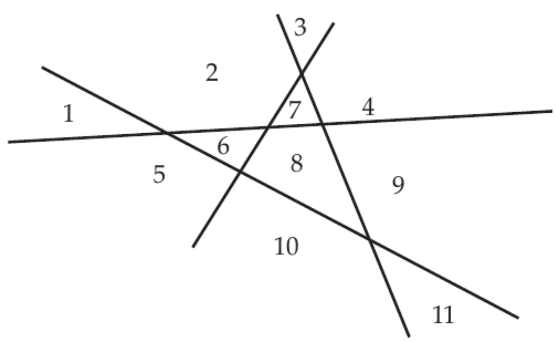

Figure 1: 4 lines in general position divide the plane into 11 regions

Solution: First we argue that r_n satisfy the recurrence

$$r_{n+1} = r_n + n + 1.$$

To see this, consider what happens to the number of regions when we draw the $n+1$st line l. This line will intersect each of the other lines at n different points, and these points partition l into $n+1$ segments, each of which divides a single region into two.

We proceed to solve the nonhomogeneous linear advancement equation

$$(A-1)r = n+1.$$

The solution to the homogeneous equation

$$(A-1)r = 0$$

are $r = c1^n = c$.

We then try to guess a particular solution, first let's try $r(n) = an + b$. Then

$$\begin{aligned}(A-1)r(n) &= (a(n+1)+b) - (an+b) \\ &= a,\end{aligned}$$

so our first guess will not work. Since doing $(A-1)$ to our function is like taking a difference quotient, if we want to end up with the nonconstant linear term $n+1$, we should try a quadratic function. So our next guess is $r(n) = an^2 + bn$. then

$$\begin{aligned}(A-1)r(n) &= \big(a(n+1)^2 + b(n+1)\big) - (an^2 + b) \\ &= 2an + a + b.\end{aligned}$$

And since we want this to be a solution, this leads to

$$2an + a + b = n + 1$$

so $a = \frac{1}{2}, b = \frac{1}{2}$. So the general solution is

$$r(n) = \frac{n^2}{2} + \frac{n}{2} + c$$

since $r(0) = 1$, we want $c = 1$, and therefore

$$r(n) = \frac{n^2+n}{2} + 1 = \binom{n+1}{2} + 1.$$

10 The Main Theorem on Linear Recurrence Equations (Ch. 9.5 in [KT17])

Theorem 10.1. *Let*

$$(A-r_1)^{k_1}(A-r_2)^{k_2}\cdots(A-r_m)^{k_m} f(n) = 0$$

be an advancement operator equation with all r_i distinct real numbers. Then all the solutions are of the form

$$f(n) = \big(c_{1,1}r_1^n + c_{1,2}nr_1^n + c_{1,3}n^2r_1^n + \ldots + c_{1,k_1}n^{k_1-1}r^n\big) + \ldots + \big(c_{1,m}r_m^n + c_{1,2}nr_m^n + c_{m,3}n^2r_1^n + \ldots + c_{m,k_m}n^{k_m-1}r^n\big)$$

We will not prove this theorem, if you are interested, a proof is in Ch 9.5 of [KT17]. Note that the theorem helps us solve any linear recurrence provided we can factor the polynomial in the advancement operator.

Example 10.2 (Example 9.21. in [KT17])**.** *Consider the advancement operator equation*

$$(A-1)^5(A+1)^3(A-3)^2(A+8)(A-9)^4 f = 0.$$

Then every solution has the form

$$\begin{aligned}f(n) = c_1 &+ c_2n + c_3n^2 + c_4n^3 + c_5n^4 \\ &+ c_6(-1)^n + c_7n(-1)^n + c_8n^2(-1)^n \\ &+ c_93^n + c_{10}n3^n + c_{11}(-8)^n \\ &+ c_{12}9^n + c_{13}n9^n + c_{14}n^29^n + c_{15}n^39^n.\end{aligned}$$

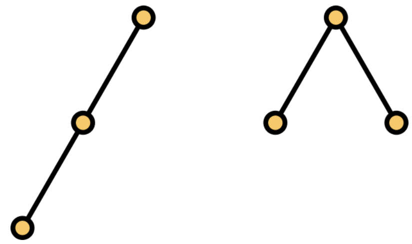

Figure 2: Two nonisomorphic rooted trees

11 Counting rooted binary ordered trees

Definition 11.1. *A tree is* ***rooted*** *if we have designated a special vertex as a root.*

We will always draw rooted trees with the root on top. For example, the two trees on figure 2 are isomorphic as graphs, but not isomorphic as rooted trees.

In a rooted tree, each vertex v has a **parent**, which is the unique vertex that is one step closer to the root than v. If w is a parent of v, we say that v is a **child** of w.

Definition 11.2. *A rooted tree is a* ***binary*** *tree if each vertex has* 0 *or* 2 *children.*

Definition 11.3. *A rooted tree is* ***ordered*** *if the children of each vertex have some ordering.*

We will usually indicate the ordering by drawing our trees in a consistent way, for example, Figure 3 shows all the rooted binary ordered trees with n leaves for $n \leq 4$

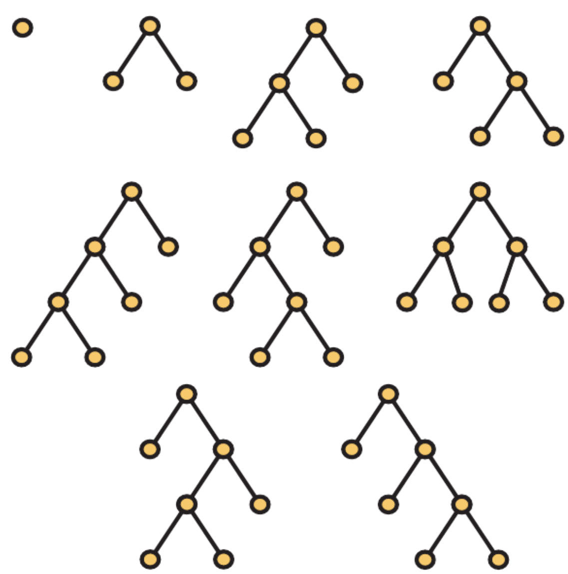

Figure 3: rooted binary ordered trees with n leaves for $n \leq 4$

Let c_n be the number of all the rooted binary ordered trees with n leaves. We see from Figure 3 that $c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5$. The structure of these trees suggest a recursive approach to counting them. For example, if $n \geq 2$, then the root must have two children. If we remove the root from the tree, these children become roots of rooted binary ordered trees with a combined number of n leaves, since the number of leaves does not change. Therefore we may pick a left child in c_k ways (for $k = 1, \ldots, n-1$), and then we may pick the right child in c_{n-k} ways. Therefore we get the recursive formula

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k}$$

if we define $c_0 = 0$, we may write this as

$$c_n = \sum_{k=0}^{n} c_k c_{n-k} \tag{3}$$

(note that this looks very similar to the Catalan recurrence, but let's ignore this for now). This is a nonlinear recurrence, so the method we developed in the previous section will not help us.

We try an approach using generating functions. Define

$$C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then $C(x)$ is the generating function associated to the counting problem. Note that the RHS of equation (3) looks very much like the coefficient in a power series squared.

Consider

$$\begin{aligned} C^2(x) &= c_0^2 + (c_0c_1 + c_1c_0)x + (c_0c_2 + c_1c_1 + c_2c_0)x^2 + \cdots \\ &= 0 + 0 + (c_0c_2 + c_1c_1 + c_2c_0)x^2 + (c_0c_3 + c_1c_2 + c_2c_1 + c_3c_0)x^3 + \cdots . \end{aligned}$$

Note that using equation 3, we see that $C^2(x)$ is almost exactly $C(x)$ again, except for the x term. We have

$$C(x) = x + C^2(x),$$

and this leads to

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2} = \frac{1 \pm (1-4x)^{1/2}}{2}.$$

We want to expand $\frac{1\pm(1-4x)^{1/2}}{2}$ into a power series. The Generalized Binomial Theorem enables us to do this, (we omit the details, if you are interested, see Ch. 9.7 of [KT17]) and we get

$$C(x) = \frac{1}{2} \pm \frac{1}{2} \mp \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^n.$$

Since the number of rooted binary ordered trees is a positive number, we may conclude that

$$C(x) = \frac{1 - \sqrt{1-4x}}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n.$$

Therefore $c_n = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}$.

References

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2, 3, 4, 6, 8