MAT344 Practice questions for the Final

Note: This is a collection of questions to help you prepare for the final, it is longer than the final will be.

1. Answer each of the following counting problems. You do not need to simplify your answer. **For this question only, you do not need to justify your answers.**

Find the coefficient of x^4 in the series expansion of

$$\frac{x}{1-2x^3}.$$

Answer

What is the number of faces of a planar graph that has 6 edges and 4 vertices?

Answer

Find the closed form (**not** the series expansion) of the generating function given by the sequence $a_n = 2n$, that is, the sequence

$$(0, 2, 4, 6, \ldots).$$

Answer

What is the value of the sum $\sum_{k=0}^{10} \binom{10}{k}$ (give the answer **as a number**)?

Answer

Find the chromatic number of the graph in Figure 1.

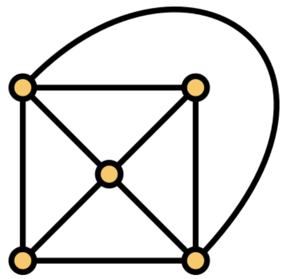

Figure 1: A graph

Answer

2. (a) Write the exponential generating function that counts the number of n-length ternary strings where each digit appears at least once.

(b) What is the coefficient of $\frac{x^n}{n!}$ in your exponential generating function?

3. Recall that for a planar graph with $|V| \geq 3$, we have $2|E| \geq 3|F|$. Show that the average degree of a planar graph is less than 6, i.e. $\dfrac{\sum_{v \in V} \deg v}{|V|} < 6$.

4. A wizard has 5 friends. During a long wizards conference, the wizard met any given friend at dinner 10 times, any given pair of friends 5 times, any given triple of friends 3 times, any given group of four friends 2 times, and all 5 friends together once. The wizard ate alone 6 times. Determine how many dinners the wizard had during the conference.

5. (a) Find the general solution to the linear homogeneous recurrence

$$f(n+3) = 5f(n+2) - 3f(n+1) - 9f(n).$$

Hint: Notice that $h(n) = (-1)^n$ satisfies this recurrence.

(b) Find the general solution to the nonhomogeneous equation

$$g(n+3) = 5g(n+2) - 3g(n+1) - 9g(n) + 8n - 4$$

6. Is the following graph planar? If so, give a planar drawing of this graph. Otherwise, prove that it is non-planar.

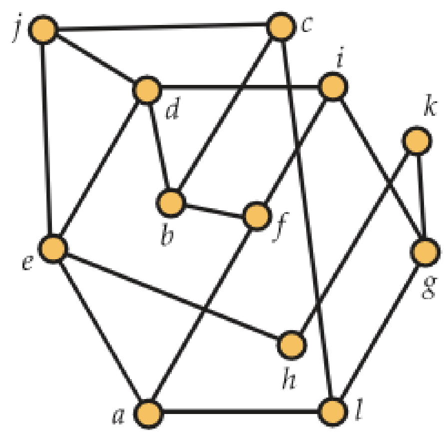

Figure 2: A graph

7. Give a proof of the identity

$$\sum_{k=1}^{n} k \cdot (n-k+1) = \binom{n+2}{3}.$$

by counting elements of a set in two different ways. **Hint:** Interpret the right hand side as triples (x, y, z) of integers with $1 \leq x < y < z \leq n+2$.

8. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

with $x_1 \geq 0$ even, $x_2 \geq 0$ and $0 \leq x_3 \leq 2$.

9. Let a_n denote the number of tilings of an $n \times 1$ rectangle using any number of blue 1×1 blocks, red 2×1 blocks or green 2×1 blocks.

(a) Give a recurrence and initial conditions for a_n.

(b) Use your recurrence and initial conditions to find a closed formula for a_n depending only on n. (If your solution to part (a) is incorrect, you will still receive marks for analysing it correctly here.)