MAT344 Practice questions for the Midterm

Note: This is a collection of questions to help you prepare for the midterm, it is much longer than the midterm will be.

1. Answer each of the following counting problems. You do not need to simplify your answer. **For this question only, you do not need to justify your answers.**

The number of ways of distributing 18 identical folders between 4 employees.

Answer

The number of edges in a complete graph on 5 vertices.

Answer

What is the number of faces of a planar graph that has 6 edges and 4 vertices?

Answer

The number of subsets of $[10] = \{1, 2, \ldots, 10\}$ that contain at least one even integer.

Answer

The number of ways of buying 8 bagels out of 5 possible flavors (the order of buying the 5 bagels does not matter).

Answer

The number of Hamilton paths in a complete graph on 7 vertices (recall that a Hamilton path is one that visits each vertex exactly once).

Answer

The number of ways of arranging three blue, two red and four green balls in a line (balls of the same color are indistinguishable).

Answer

The number of ways the score of an ice hockey game can go to a 4-4 tie if the home team is never behind in goals throughout the game

Answer

The number of binary strings of length 8 that contain at least two 1s.

Answer

The number of ways of rearranging the letters of the word "TORONTO" (they do not need to be English words).

Answer

The minimal number of socks that an (eight-legged) octopus needs to pull out blindly from their drawer to make sure they have a matching set of 8 socks if they have a red, blue, green, yellow, grey and white socks in their drawer.

Answer

The number of lattice paths from $(0, 0)$ to $(4, 4)$ that do not go above the diagonal $y = x$ (recall that lattice paths consist entirely of up and right steps).

Answer

2. Give a **combinatorial proof** of the identity:

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k},$$

where n is a positive integer.

3. Give a **combinatorial proof** of the identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

where n is any positive integer.

4. Suppose that f is a function defined on the nonnegative integers such that $f(0) = 0$ and $f(n) = n^2 + f(n-1)$. By induction, or otherwise, prove that

$$f(n) = \frac{n(n+1)(2n+1)}{6}.$$

5. By induction, or otherwise, prove that

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1,$$

where F_i denotes the ith Fibonacci number (i.e. $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

6. We draw n mutually intersecting circles in the plane in a way that each one crosses each other one exactly twice and no three intersect in the same point. (As examples, you can think of Venn diagrams with two or three mutually intersecting sets) Find a recurrence relation for the number $r(n)$ of regions into which the plane is divided by n circles.

7. Find a recurrence relation for the number of binary strings that do not contain exactly two 1s in a row (to clarify, for example, the string 1011110 is allowed but 01100111 is not).

8. Find the number of nonnegative integer solutions to the following problems

   (a) The equality

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

   (b) The inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 100,$$

9. Figure 1 shows a proposed network of computers.

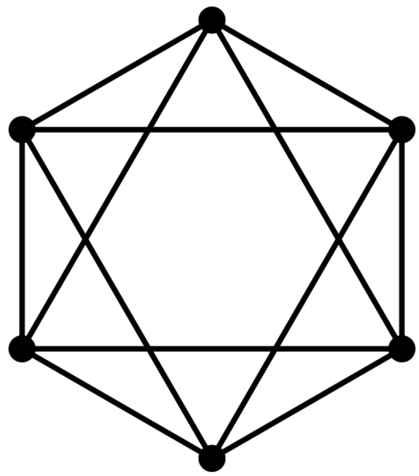

Figure 1: A network

The lines show which computers can communicate directly with which others. Consider two ways of assigning computers to the nodes of the network different if there are two computers that communicate directly in one assignment and that don't communicate directly in the other. In how many different ways can we assign six distinguishable computers to the network?

10. Recall that an **Euler circuit** in a multigraph G is a walk that traverses every edge exactly once and arrives back at the starting vertex. Prove that if G is any connected multigraph on n vertices, you can always add at most $\lfloor \frac{n}{2} \rfloor$ new edges in a way that the resulting graph has an Euler circuit. You may use theorems from lecture without proving them.

11. Fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets S_i of $[n] = \{1, 2, \ldots, n\}$ subject to each of the following conditions **separately**, that is, the three parts are independent problems.

(a) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$.

(b) The S_i are pairwise disjoint (i.e. $S_i \cap S_j = \emptyset$ for $i \neq j$).

(c) $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$.