MAT344 Problem Set 10
(due Thursday, Dec/5, noon)

Notes:

- For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.
- When submitting to Crowdmark, please ensure that your uploads are *legible*, *correctly rotated*, and *properly matched with the correct problems.* Any improperly uploaded problem scans will not be graded.
- Any assignments submitted after the deadline will not be accepted.

Part A

Three randomly chosen questions from this part will be marked.

Problem 1. Consider the recurrence relation $r_{n+2} = r_{n+1} + 3r_n$.

1. What is the general solution to the recurrence relation?
2. Provide a closed-form solution to the recurrence relation when $r_0 = 1$ and $r_2 = 5$ (note that we specified r_2 but not r_1).

Problem 2. Find the general solution of the nonhomogeneous advancement operator equation

$$(A - 5)(A + 2)f(n) = 3^n$$

Problem 3. Find the general solution of the advancement operator equation

$$(A^3 + 3A^2 + 3A + 1)f(n) = 0.$$

Problem 4. Find and solve a recurrence equation for the number g_n of ternary strings of length n that do not contain 102 as a substring.

Figure 1: The 1×1 tile and L-tiles

Part B

Two randomly chosen questions from this part will be marked.

Problem 5. Let t_n denote the number of ways to tile a $2 \times n$ rectangle using 1×1 tiles and L-tiles. L tiles are 2×2 tiles with one of the squares missing. Figure 1 shows the L tiles in all possible rotations.

1. Find a recursive formula for t_n, including the appropriate initial conditions. **Hint**: there are 7 cases you need to consider to reduce a $2 \times n$ rectangle to a smaller rectangle, and 3 initial conditions.

2. Find a closed formula for t_n.

Problem 6. There is a famous puzzle called the Towers of Hanoi that consists of three pegs and n circular disks, all of different sizes. The disks start on the leftmost peg, with the largest disk on the bottom, the second largest on top of it, and so on, up to the smallest disk on top. The goal is to move the disks so that they are stacked in this same order on the rightmost peg. However, you are allowed to move only one disk at a time, and you are never able to place a larger disk on top of a smaller disk. Let t_n denote the fewest moves (a move being taking a disk from one peg and placing it onto another) in which you can accomplish the goal. Determine an explicit formula for t_n.

Problem 7. Consider the recurrence

$$d_n = nd_{n-1} + (-1)^n.$$

Let $d(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$ be the exponential generating function for d_n.

1. Find the exponential generating functions for the following sequences
   (a) nd_{n-1} (**Hint:** Try $xd(x)$),
   (b) $(-1)^n$,
   (c) $n!$.

2. Use the recurrence to prove that $d(x) = \frac{e^{-x}}{1-x}$.

3. Find the coefficient of $\frac{x^n}{n!}$ in $d(x)$. Do you recognize this number?

Part C

This question will be marked for completion only.

Problem 8. Describe a recurrence that has a closed form solution you can find using tools of this chapter.