MAT344 Problem Set 2
(due Thursday, Sept 26, noon)

Notes:

1. For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.
2. When submitting to Crowdmark, please ensure that your uploads are *legible*, *correctly rotated*, and *properly matched with the correct problems*. Any improperly uploaded problem scans will not be graded.
3. Any assignments submitted after the deadline will not be accepted.

Part A

Three randomly chosen questions from this part will be marked.

Problem 1. How many integer solutions are there to the following inequality:

$$a_1 + a_2 + a_3 + a_4 + a_5 \leq 1729$$

if $a_2, \ldots, a_5$ must all be nonnegative, and $a_1 \geq 10$?[1]

Problem 2. After expanding $(a + b + c + d)^7$ and combining like terms, how many terms are there? Justify your answer without performing the expansion.

Problem 3. Hockey games are played over three periods, and ties are broken by one overtime period. Suppose the Leafs tied the Canadiens 5-5 after regulation time. In how many ways could the score change over the game if the Leafs were never losing?

Problem 4. What is the coefficient of x^5y^2 in $(x + y + 1)^9$?

[1] 1729 is a special integer, known as the *Hardy-Ramanujan number*. It comes from the following anecdote: British mathematician G. H. Hardy was visiting the Indian mathematician Srinivasan Ramanujan (who was known for his preternatural facility with numbers) in hospital. He related their conversation:

> I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. “No,” he replied, “it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.”

Source: https://en.wikipedia.org/wiki/1729_(number)

Part B

Two randomly chosen questions from this part will be marked.

Remember: to give a combinatorial proof, you have to exhibit the set of objects you are counting and explicitly describe the two ways you are counting them.

Problem 5. Let n be a positive integer. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}.$$

Problem 6. Let m, n be positive integers. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}.$$

Problem 7. You want to walk 4 blocks North and 6 blocks East in a city with a grid of East-West and North-South streets.

(a) In how many ways can you do this?

(b) The intersection 2 blocks North and 2 blocks East (at $(2, 2)$) from your starting point is blocked, so you can not pass through it. In how many ways can you walk to your destination now?

(c) Find the most inconvenient place for the block on the (East-West) street 2 streets North from your starting point, i.e. find the intersection $(k, 2)$ which, if blocked, results in the least number of available paths to your destination.

(d) Consider the intersection you found in the previous part. By moving it along the (North-South) street (to some other intersection (k, j)), can you make it more inconvenient?

(e) Can we conclude that the block at the most inconvenient place in the entire grid? Why or why not?

(f) **Challenge question (will not be graded):** Repeat parts (a)-(c) in the case where you have to walk 4 blocks North and n blocks East. *Hint: try to use Calculus.*

Part C

This question will be marked for completeness only

Problem 8. Give an example of a counting problem that can be solved using binomial coefficients.