MAT344 Problem Set 3
(due Thursday, October 3, noon)

Notes:

- For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.
- When submitting to Crowdmark, please ensure that your uploads are *legible*, *correctly rotated*, and *properly matched with the correct problems.* Any improperly uploaded problem scans will not be graded.
- Any assignments submitted after the deadline will not be accepted.

Remember: To give a recursive formula you have to relate values of a certain function $f(n)$ defined over the natural numbers to values $f(k)$ of the functions with $k < n$.

Part A

Three randomly chosen questions from this part will be marked.

Problem 1. Give a recursive formula for the number of ways P_n of parenthesizing n factors. For example, here are all the ways of parenthesizing 4 factors:

$$a(b(cd)), a((bc)d), (ab)(cd), ((ab)c)d, (a(bc))d$$

Problem 2. Consider a $1 \times n$ checkerboard. The squares of the checkerboard are to be painted white and gold, but no two consecutive squares may both be painted white. Let $p(n)$ denote the number of ways to paint the checkerboard subject to this rule. Find a recursive formula for $p(n)$ valid for $n \geq 3$.

Problem 3. Let f be a function on the nonnegative integers such that $f(0) = 0$ and $f(n) = n + f(n-1)$. Use mathematical induction to prove that $f(n) = \frac{n(n+1)}{2}$.

Problem 4. There are n hungry lions on an island and a piece of meat. The meat is poisoned, and any lion eating it will fall asleep. A lion who is asleep because of the poison may be eaten by another lion, but the poison stays potent and will still cause that lion to fall asleep at which point they may be eaten by another lion (and so on, arbitrarily many times). The first lion has a choice of either eating the meat or not. They are hungry, but they would prefer to stay hungry over being eaten. Should they eat the meat? **Hint:** Think recursively. What does this problem have to do with n?

Part B

Two randomly chosen questions from this part will be marked.

Problem 5. Find a combinatorial proof of the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_{n+1},$$

where F_n denotes the n-th Fibonacci number. **Hint:** Think about the domino tiling interpretation of F_n discussed in lecture.

Problem 6. Use induction to prove that for all integers $n \geq 1$, the quantity

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9.

Problem 7. A $2 \times n$ checkerboard is to be tiled using three types of tiles. The first tile is a white 1×1 square tile. The second tile is a red 2×2 tile and the third one is a black 2×2 tile. Let $t(n)$ denote the number of tilings of the $2 \times n$ checkerboard using white, red and black tiles.

(a) Find a recursive formula for $t(n)$ and use it to determine $t(7)$.

(b) Let $f(n) = c_1 2^n + c_2(-1)^n$. Determine c_1 and c_2 so that $f(0) = f(1) = 1$.

(c) Prove that $f(n)$ satisfies the same recurrence relation as $t(n)$.

(d) Can we now conclude that $f(n) = t(n)$ for all positive integers n?

Part C

This question will be marked for completion only.

Problem 8. Give an identity which can be proved by induction or by a combinatorial argument. Which do you prefer?