MAT344 Problem Set 6
(due Saturday, Nov 2, noon)

Notes:

- For all the questions, always *explain your reasoning* and refer to the results you are using. Just a number (even if it is the correct final answer) will **not** get you full credit.
- When submitting to Crowdmark, please ensure that your uploads are *legible*, *correctly rotated*, and *properly matched with the correct problems*. Any improperly uploaded problem scans will not be graded.
- Any assignments submitted after the deadline will not be accepted.

Part A

Three randomly chosen questions from this part will be marked.

Problem 1. Find a minimal spanning tree of the graph in Figure 1.

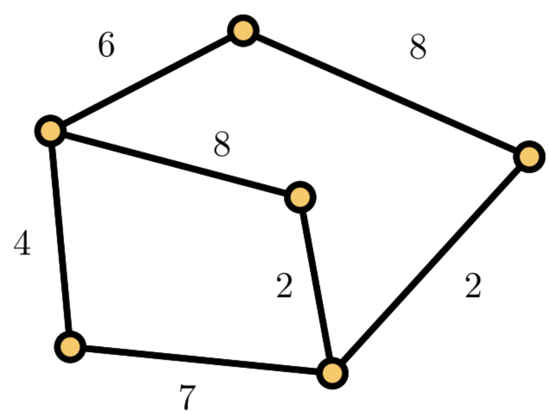

Figure 1: A weighted graph

Problem 2. Prove that every planar graph G has a vertex v with $\deg_G(v) \leq 5$.

Problem 3. For the next Olympic Winter Games, the organizers wish to expand the number of teams competing in the sport of curling. Their goal is to divide 14 teams into two groups each. The organizers would like that in the first round, each team from a group will play a total of seven games against distinct opponents: five from their own group and two from the other group.

They're having trouble setting up such a schedule, so they've come to you. By using an appropriate graph-theoretic model, either argue that they cannot use their current plan or devise a way for them to do so.

Problem 4. A new local bank is being created and will establish a headquarters h, two branches b_1 and b_2, and four ATMs a_1, a_2, a_3, a_4. They need to build a computer network such that the headquarters, branches, and ATMs can all intercommunicate. Furthermore, they will need to be networked with the Federal Reserve Bank, f. The costs of the feasible network connections (in units of \$10,000) are listed below:

hf	80	hb_1	10	hb_2	20	b_1b_2	8
fb_1	12	fa_1	20	b_1a_1	3	a_1a_2	13
ha_2	6	b_2a_2	9	b_2a_3	40	a_1a_4	3
a_3a_4	6						

The bank wishes to minimize the cost of building its network (which must allow for connection, possibly routed through other nodes, from each node to each other node), however due to the need for high-speed communication, they must pay to build the connection from h to f as well as the connection from b_2 to a_3. Give a list of the connections the bank should establish in order to minimize their total cost, subject to this constraint. Be sure to explain how you selected the connections and how you know the total cost is minimized.

Part B

Two randomly chosen questions from this part will be marked.

Problem 5. Show that the average degree of a planar graph is less than 6, i.e. $\dfrac{\sum_{v\in V} \deg v}{|V|} < 6$.

Problem 6.

Definition. *A* ***cut*** *in a network is a pair* (X, Y) *of subsets of the vertex set* V *such that* $X \cup Y = V, s \in X, t \in Y$. *The* ***capacity*** $c(X, Y)$ *of the cut is the sum of the capacities of the edges directed from* X *to* Y *(i.e. edges* $e = (x, y)$ *with* $x \in X$ *and* $y \in Y$*).*

Figure 2 shows a cut in a network.

Let f be a flow in a network and let $V = X \cup Y$ be a cut. The **strength** of the flow is defined as the total amount of flow leaving the source

$$|f| = \sum_{y\in V} f(s, y).$$

(a) Use the conservation laws to prove that

$$|f| = \sum_{x\in X} \left[\sum_{y\in V} f(x, y) - f(y, x) \right].$$

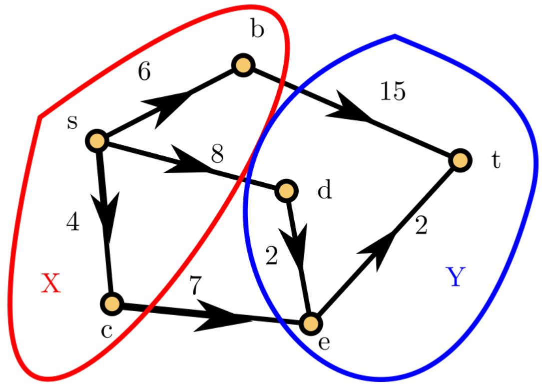

Figure 2: A cut in a network

(b) Use the above equality to prove that the capacity $c(X, Y)$ of the cut is an upper bound to the strength of the flow.

Problem 7. Let G be a connected weighted graph with distinct edge weights. Let e_{max} be the edge with maximum weight and e_{min} the edge with minimum weight. For each of the following claims, either prove the claim, or provide a counterexample.

(a) Every minimum spanning tree of G must contain e_{min}

(b) If e_{max} is in a minimal spanning tree, then removing it from G must disconnect G

(c) No minimum spanning tree contains e_{max}

(d) G has a unique minimal spanning tree

(e) $e_{max} \neq e_{min}$

Part C

This question will be marked for completion only.

Problem 8. Give an example of a real-world problem that can be solved by Prim's algorithm.