Tutorial 2

Binomial Patterns

Learning Objectives

In this tutorial you will be finding and proving patterns in the parity of binomial coefficients. The arrangement of binomial coefficients in a triangle was known long before Blaise Pascal wrote about it, and is variously known as Yang Hui's triangle, the Khayyam triangle, Tartaglia's triangle or the Staircase of Mount Meru.

These problems relate to the following course learning objectives: *Describe solutions to iterated processes by relating recurrences to combinatorial identities*, and *prove combinatorial identities by counting a set of objects in two ways.*

1 Even and Oddness

Row n of the triangle contains the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Using the identities $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$, we can construct the terms of each line by adding two terms from the previous line. This triangle contains many patterns when colouring all coefficients divisible by a given prime or 4, 8 or 9, but other composites are unknown.

1. Write the first ten rows of the binomial triangle, from $n = 0$ to $n = 9$, and replace each $\binom{n}{i}$ with 0 if it is even or 1 if it is odd. Explain how to get the next line from a previous line of 1s and 0s.
2. Which lines have only odd numbers? Which lines have only even numbers, aside from $\binom{n}{0}$ and $\binom{n}{n}$?
3. For each row, write the binary expansion of n, and the binary expansion of every i where $\binom{n}{i}$ is even. What do you notice?
4. Show that if n is $a_k a_{k-1} \ldots a_1 a_0$ in binary and i is $b_k b_{k-1} \ldots b_1 b_0$ (possibly with leading zeros), then $\binom{n}{i}$ is odd if and only if $b_m \leq a_m$ for every binary digit. (Hint: show $(1+x)^2$ and $(1+x^2)$ have the same parity, then expand $(1+x)^n$ in binary).

Base 3

Instead of considering even or oddness, we can ask about the remainder of $\binom{n}{i}$ after dividing by 3. This replacement will give a triangle with three types of entries: $0, 1$, or 2. These relate to ternary expansions, $n = a_k 3^k + \cdots a_0 3^0$.

5. Replace the binomial coefficients with their remainders after dividing by 3 to construct a triangle of 0s, 1s and 2s. Explain how to get the next line from a previous line of 0s, 1s and 2s.
6. Which lines have only 0s, aside from $\binom{n}{0}$ and $\binom{n}{n}$? Prove the binomial coefficients (aside from $\binom{n}{0}$ and $\binom{n}{n}$) in these lines are all divisible by 3. (Harder: prove only these lines have this property).
7. Make a statement similar to question 4 for this triangle.

1. The usual triangle looks like

$n=0$:										1										
$n=1$:									1		1									
$n=2$:								1		2		1								
$n=3$:							1		3		3		1							
$n=4$:						1		4		6		4		1						
$n=5$:					1		5		10		10		5		1					
$n=6$:				1		6		15		20		15		6		1				
$n=7$:			1		7		21		35		35		21		7		1			
$n=8$:		1		8		28		56		70		56		28		8		1		
$n=9$:	1		9		36		84		126		126		84		36		9		1	

And the even/odd triangle looks like

$n=0$:										1									
$n=1$:									1		1								
$n=2$:								1		0		1							
$n=3$:							1		1		1		1						
$n=4$:						1		0		0		0		1					
$n=5$:					1		1		0		0		1		1				
$n=6$:				1		0		1		0		1		0		1			
$n=7$:			1		1		1		1		1		1		1		1		
$n=8$:		1		0		0		0		0		0		0		0		1	
$n=9$:	1		1		0		0		0		0		0		0		1		1

The next line has a 0 between two numbers on the previous line if they are equal, and a 1 if they are different.

2. Lines where $n = 2^k - 1$ are all odd, and lines where $n = 2^k$ are all even. It is easier to show that $\binom{2^k}{i}$ is always even, by considering the powers of 2 dividing $2^k!$ and $i!(2^k - i)!$. Then the previous row must be all 1s, by using the rule for generating rows.

3. As an example, we have $5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, so 5 is 101 as a binary string. The digits where $\binom{5}{i}$ is even are $i = 2$ and $i = 3$, which are 010 and 011. Notice that each have a 1 in the position where 5 has a 0.

4. Write $\equiv$ to mean same parity. Using $(1+x)^2 \equiv (1+x^2)$, we have $(1+x)^{2^k} \equiv (1+x^{2^k})$, so if the binary expansion of n is $a_k \dots a_1 a_0$, then

$$(1+x)^n \equiv (1+x^{2^k})^{a_k} \cdots (1+x^2)^{a_1}(1+x)^{a_0}.$$

We know the coefficient of x^i is $\binom{n}{i}$, so it is $\equiv 1$ on the right side if every $b_m = 1$ in the binary expansion of i also has $a_m = 1$, so that it appears on the right side.

5. The base 3 triangle looks like

$$\begin{array}{l ccccccccccccccccccc}
n=0: & & & & & & & & & & 1 & & & & & & & & & \\
n=1: & & & & & & & & & 1 & & 1 & & & & & & & & \\
n=2: & & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
n=3: & & & & & & & 1 & & 0 & & 0 & & 1 & & & & & & \\
n=4: & & & & & & 1 & & 1 & & 0 & & 1 & & 1 & & & & & \\
n=5: & & & & & 1 & & 2 & & 1 & & 1 & & 2 & & 1 & & & & \\
n=6: & & & & 1 & & 0 & & 0 & & 2 & & 0 & & 0 & & 1 & & & \\
n=7: & & & 1 & & 1 & & 0 & & 2 & & 2 & & 0 & & 1 & & 1 & & \\
n=8: & & 1 & & 2 & & 1 & & 2 & & 1 & & 2 & & 1 & & 2 & & 1 & \\
n=9: & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1
\end{array}$$

6. The lines with all 0s are powers of 3: $1, 3, 9, 27, \ldots$. Again, this can be shown by considering the largest power of 3 dividing the numerator and denominator in $\binom{3^k}{i}$. To show only these are all 0, show that 3 does not divide $\binom{n}{3^k}$ when 3^k is the largest power of 3 dividing n.

7. We can generalize the previous argument to Lucas's theorem:

$$\binom{n}{i} \equiv \binom{a_k}{b_k} \cdots \binom{a_1}{b_1}\binom{a_0}{b_0} \pmod{p}$$

for any prime divisor. The situation for composites is more complicated, because we don't have $(1+x)^c \equiv (1+x^c)$ if c is not a prime power.