# The Volume of the Unit $n$-Ball

Let $V_n(r)$ denote the volume of a ball of radius $r$ in $\mathbb{R}^n$. Clearly:

$$V_n(r) = C_n r^n$$

where $C_n$ is the volume of the unit $n$-ball.

It is not difficult to find a recurrence relation on the $C_n$. In particular, let $D$ be the unit disc in $\mathbb{R}^2$. Then:

$$\begin{aligned} C_{n+2} &= \int_D V_n(\sqrt{1-r^2})\, dr\, d\theta \\ &= \int_0^1 2\pi r\, V_n(\sqrt{1-r^2})\, dr \\ &= \int_0^1 2\pi r\, C_n\, (1-r^2)^{n/2}\, dr \end{aligned}$$

This last integral can be evaluated using a straightforward $u$-substitution (namely $u = 1 - r^2$):

$$\begin{aligned} C_{n+2} &= \int_0^1 \pi\, C_n\, u^{n/2}\, du \\ &= \frac{\pi}{1+n/2} C_n \end{aligned}$$

Since $C_2 = \pi$, we get:

$$C_{2n} = \frac{\pi^n}{n!}$$

Since $C_1 = 2$ (i.e. the area of a 1-ball is $2r$), we get:

$$C_{2n+1} = \frac{2 \cdot \pi^n}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n+1}{2}}$$

We can put these in particularly nice form:

$$C_{2n} = \frac{(2\pi)^n}{2 \cdot 4 \cdot 6 \cdots (2n)} \quad \text{and} \quad C_{2n+1} = \frac{2\,(2\pi)^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$