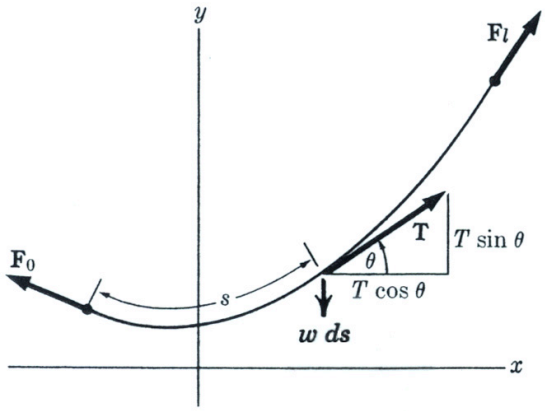
Problem. (MAE/TAM) The Shape of a Hanging Cable.

The figure below shows part of a flexible cable hanging under its own weight (which is w N/m). The figure also shows a diagram of the forces which, in equilibrium, must add to zero. A flexible cable is one that cannot support bending or shear forces (or it would just deform) and so only the tension T along a local tangent to the cable enters the problem.



In the Appendix to this workshop, the differential equation for the shape of the cable implied by the force balance is derived. Do not read it now but if you are interested in mechanical engineering or just interested, you should look at it later. Let $y(x)$ be the function whose graph represents the shape of the cable. The differential equation for $y(x)$ is:

$$\frac{d^2y}{dx^2} = \frac{w}{C}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{1}$$

where C is a constant associated with the horizontal component of the tension.

a) Integrate (1) once and solve the resulting equation for dy/dx. You should have an integration constant A. (**Hint:** Write dy/dx as u, so that d^2y/dx^2 becomes du/dx, and solve the differential equation for u.)

Solution. Writing $u = dy/dx$, (1) becomes

$$\frac{du}{dx} = \frac{w}{C}\sqrt{1+u^2}.$$

Now we separate variables and integrate separately

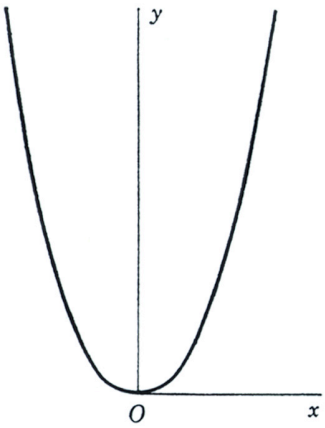
$$\begin{aligned}\frac{du}{\sqrt{1+u^2}} &= \frac{w}{C}\,dx\\ \int \frac{du}{\sqrt{1+u^2}} &= \int \frac{w}{C}\,dx\\ \sinh^{-1}(u) &= \frac{w}{C}\,x + A\end{aligned}$$

The last step comes from recognizing the form of the derivative of $\sinh^{-1}$.

Thus $\boxed{dy/dx = \sinh(wx/C + A)}$.

b) Using a coordinate system with the origin placed at the lowest point of the curve (as shown below), find A.

(**Hint:** Where in the figure can you determine dy/dx exactly?)



¿From part **a)**, we have the equation

$$\frac{dy}{dx} = \sinh\left(\frac{w}{C}\,x + A\right).$$

The coordinate system to the left shows that at $x = 0$, $dy/dx = 0$. Thus we have

$$0 = \sinh(0 + A).$$

That is, $0 + A = \sinh^{-1}(0)$, or $\boxed{A = 0}$.

c) Integrate the new equation from parts **a)** and **b)** to find the shape of the cable. In addition to the parameters w and C, there should be a new constant called B. Using the figure above, find B. This shape is important enough to have a name: the *catenary*.

Solution. Now we separate the variables x and y:

$$dy = \sinh\left(\frac{w}{C}\,x\right)\,dx$$
$$\int dy = \int \sinh\left(\frac{w}{C}\,x\right)\,dx$$
$$y = \frac{C}{w}\cosh\left(\frac{w}{C}\,x\right) + B$$

In order to have $y = 0$ at $x = 0$ (as shown in the figure), we must have

$$0 = \frac{C}{w}\cdot 1 + B,$$

or $B = -C/w$. Thus the equation of the catenary in these coordinates is

$$\boxed{y(x) = \frac{C}{w}\left[\cosh\left(\frac{w}{C}\,x\right) - 1\right]}.$$

d) If the cable is symmetric about the origin and attached at end points located at $x = -L$ and $x = +L$, find an integral expression for the length, ℓ, and evaluate it.

Solution. From the arclength formula $\ell = \int_{-L}^{+L} \sqrt{1 + (y'(x))^2}\,dx$, we find

$$\ell = \int_{-L}^{+L} \sqrt{1 + \sinh^2\left(\frac{w}{C}\,x\right)}\,dx = \int_{-L}^{+L} \cosh\left(\frac{w}{C}\,x\right)\,dx$$
$$= 2\int_{0}^{L} \cosh\left(\frac{w}{C}\,x\right)\,dx = \boxed{2\frac{C}{w}\sinh\left(\frac{wL}{C}\right)}.$$