1. Introduction

The goal of this course is to give an introduction to perverse sheaves, to prove the decomposition theorem and then to highlight several applications of perverse sheaves in current mathematics.

The story of perverse sheaves starts with, in some sense, the work of Goresky- MacPherson [GM80] and their construction of intersection homology. This is a homology theory for singular spaces that also has Poincaré duality. Goresky-MacPherson [GM83] also show that there is a complex of sheaves called the intersection cohomology or IC sheaf such that when one takes the hypercohomology, one obtains the intersection homology of the the space.

The work of Belinson-Bernstein-Deligne-Gabber [BBD] gives a more intrinsic construction of the IC sheaves and gives rise to a more general class which they call perverse sheaves.

2. Stratifications

Definition 2.1. A *stratification* of a topological space X is a choice of P, a partially ordered set with $|P| < \infty$, and of subspaces X_i, called stratum, for each $i \in P$ satisfying the following list of axioms:

1. Each X_i is a topological manifold.
2. $X_i \cap X_j = \emptyset$ if $i \neq j$.
3. X_i is locally closed.
4. $\cup_{i \in P} X_i = X$.
5. If $X_i \cap \bar{X}_j \neq \emptyset$ then $X_i \subset \bar{X}_j$. This relation is equivalent to $i \leq j$.

Example 2.2. The locus of $xy = 0$ in $\mathbb{C}^2$ has a stratification consisting of the origin as one stratum and the remainder of the variety as the other. A second stratification consists of the punctured x-axis, the punctured y-axis and the origin.

Note 2.3. It is common to assemble all strata of a fixed dimension together into one space. In this case the poset P becomes linear and the strata are typically indexed by their dimension. For a connected space, the closure of the top dimensional stratum is thus dense in space and is typically referred to as the open or dense stratum.

Note 2.4. The closure of a stratum may still be smooth. moreover taking iterated singular part does give a stratification, but as we will see below, it has some problems.

2.1. Whitney conditions.

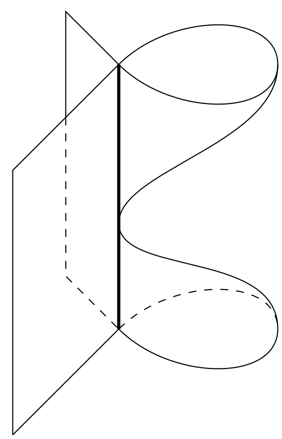
The following two conditions on a pair X, Y of locally closed submanifolds of $\mathbb{R}^n$ of dimensions i and j were defined by Whitney:

- X and Y satisfy condition A if for any sequence $x_1, x_2, \cdots \in X$, converging to $y \in Y$ with tangent i-planes covering to an i-plane T, then T contains the tangent plane to y in Y.

- X and Y satisfy condition B if for any sequence of points $x_1, x_2, \cdots \in X$ and $y_1, y_2, \cdots \in Y$ converging to $y \in Y$ such that the secant lines between x_i and y_i converge to a line L then L is in T as described above.

A stratification whose strata satisfy the Whitney conditions is called a *Whitney stratification.*

Example 2.5. Consider the surface $y^2 - x^3 - z^2x^2 = 0$:



A natural stratification might be to take the z-axis (bold in the above diagram) as one strata and then have the remaining smooth part as the other strata. This violates Whitney B. A Whitney stratification would have to have the origin as a third strata.

Note 2.6. Every complex variety has a Whitney stratification with all even dimensional strata.

2.2. **Locally Trivial Stratifications.**

Definition 2.7. A stratification is called *locally trivial* if for each $x \in X_i$, there is a neighbourhood U of x in X, an open ball B around x in X_i and a stratified space L such that there exists a homeomorphism $\phi : B \times CL \to U$ preserving the stratification. Here CL is the open cone over L.

Thom [Thom69] and more recently Mather [Mather70] show that any Whitney stratification is also locally trivial stratification.

Note 2.8. Depending on the author the definition of stratified space often contains one of the previous two conditions as well.

Theorem 2.9 ([Goresky78])**.** *Any locally trivial stratified space X is triangulable.*

3. Intersection Homology

3.1. **Borel-Moore homology.** The Borel-Moore homology of a space X, denoted $H_\bullet^{BM}(X)$ is to normal homology what standard cohomology is to cohomology with compact supports.

There are several equivalent definitions of $H_\bullet^{BM}(X)$, for now the most convenient is in terms of singular simplicies. Let $C_i^{BM}(X)$ be the be the chain complex of infinite singular chains which are locally finite. By locally finite we mean that for any compact set $D \subset X$ and $\eta \in C_i^{BM}(X)$, only finitely many simplicies in η meet D. The singular boundary map is still well defined and $H_\bullet^{BM}(X)$ is the homology of this complex.

The support of $\eta \in H_i^{BM}(X)$ is simply the union of the supports of the simplicies in η and will be denoted $|\eta|$.

Lemma 3.1. *$|\eta|$ is closed for $\eta \in H_i^{BM}(X)$.*

Proof. This is due to the local finiteness condition. □

For this reason Borel-Moore homology is often referred to as homology with closed supports and if we restrict to Borel-Moore chains with compact support, we obtain the singular homology of the space which is sometimes referred to as homology with compact supports

Note 3.2. For X a compact space, $H_\bullet^{BM}(X) = H_\bullet(X)$.

Theorem 3.3 (Poincaré Duality)**.** *If X is an oriented n-manifold, the intersection pairing*

$$H_i^{BM}(X) \times H_{n-i}(X) \to H_0(X) \to \mathbb{C}$$

is a perfect pairing.

Example 3.4. For X and Y, the loci of $x^2 + y^2 = 1$ and $x^2 + y^2 = z^2$, we compute the Borel-Moore and standard homology:

i	$H_i^{BM}(X)$	$H_i(X)$
0	0	$\mathbb{C}$
1	$\mathbb{C}$	$\mathbb{C}$
2	$\mathbb{C}$	0

i	$H_i^{BM}(Y)$	$H_i(Y)$
0	0	$\mathbb{C}$
1	$\mathbb{C}$	0
2	$\mathbb{C} \oplus \mathbb{C}$	0

For the singular variety we see that Poincaré duality fails. In particular it fails since any 1-chain representing the generator of $H_1^{BM}(Y)$ cannot be made transverse to the singular point. Goresky-MacPherson proposed fixing this by discarding all chains whose support intersects the singular locus in the incorrect dimension.

In the above example, this discards every 1 or 0-chain whose support intersects the origin. The homology of the resulting chain complex is the intersection homology we we have

i	$IH_i(Y)$	$IH_i^c(Y)$
0	0	$\mathbb{C} \oplus \mathbb{C}$
1	0	0
2	$\mathbb{C} \oplus \mathbb{C}$	0

3.2. **Perversities.** In order to define the intersection homology of a space X in general, we will need to take into account not only the singular set of the space, but also the singular set of the singular set etc. We will pick a stratification of X and then we will need to determine how chains will be allowed to intersect the various strata. This will be determined by a *perversity*, which will set the allowed defect in the intersection of a chain with a stratum.

Definition 3.5. A *perversity* of dimension n is map $p : \{2, 3, \dots, n\} \to \mathbb{Z}_{\geq 0}$ such that $p(2) = 0$ and $p(k+1) - p(k) \in \{0, 1\}$.

There are 4 important perversities to consider

$$\begin{aligned} \underline{0} &= (0, 0, 0, \dots, 0) && \text{0-perversity} \\ \underline{m} &= (0, 0, 1, 1, 2, 2, \dots) && \text{lower middle perversity} \\ \underline{n} &= (0, 1, 1, 2, 2, 3, \dots) && \text{upper middle perversity} \\ \underline{t} &= (0, 1, 2, 3, \dots, n-2) && \text{top perversity} \end{aligned}$$

moreover two perversities are complementary if $\underline{p} + \underline{q} = \underline{t}$.

Definition 3.6. An i-chain η is said to be $\underline{p}$*-allowable* if for each $k \geq 2$ (real) co-dimensional strata X_α of X, we have

1. $\dim_{\mathbb{R}}(|\eta| \cap X_\alpha) \leq i - k + p(k)$
2. $\dim_{\mathbb{R}}(|\delta\eta| \cap X_\alpha) \leq i - 1 - k + p(k)$

Note that the expected dimension of intersection of X_α and $|\eta|$ is exactly $i - k$, so $p(k)$ actually giving the defect from the 'correct' intersection dimension.

Definition 3.7. Given a perversity $\underline{p}$, let $C_i^{\underline{p}}(X)$ be the collection of $\underline{p}$-allowable Borel-Moore i-chains.

Lemma 3.8. *The boundary map δ is well defined, in particular, if $\eta \in C_i^{\underline{p}}(X)$, then $\delta\eta \in C_{i-1}^{\underline{p}}(X)$*

Proof. Since η is $\underline{p}$-allowable, for any strata X_α of codimension k we have

$$\dim_{\mathbb{R}}(|\delta\eta| \cap X_\alpha) \leq i - 1 - k + p(k).$$

moreover $\delta\delta\eta = 0$, so we see that the $i-1$ Borel-Moore chain $\delta\eta$ is $\underline{p}$-allowable. □

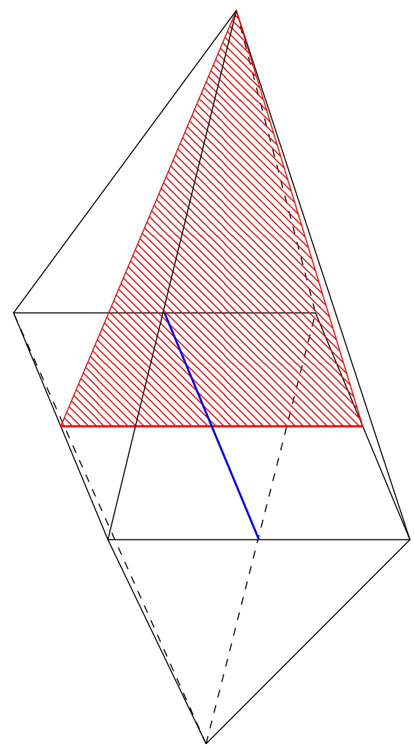
Definition 3.9. Given a stratified space X and a perversity $\underline{p}$, the *intersection homology* $I^{\underline{p}}H_i(X)$ is the i-th homology of the complex $C_\bullet^{\underline{p}}$.

Note 3.10. If X has no strata of co-dimension i, then for a perversity $p(i)$ doesn't matter. Hence for a complex variety with only even dimensional strata, the two middle perversities give the same intersection homology. This is what is typically referred to as the intersection homology of a complex variety.

Theorem 3.11 ([GM80]). *The spaces $I^{\underline{p}}H_\bullet(X)$ do not depend on the stratification of X.*

Example 3.12. Consider the suspension X of the standard 2 torus. Its homology is $H_0(X) = \mathbb{C}$, $H_1(X) = 0$, $H_2(X) = \mathbb{C} \oplus \mathbb{C}$ and $H_3(X) = \mathbb{C}$. Since this is a compact space, $H_\bullet^{BM}(X)$ is the same. The singular locus Σ consists of two points, the top and bottom of the suspension. $X - \Sigma$ is smooth, so X is has a Whitney stratification consisting of Σ and $X - \Sigma$. Since X has dimension 3, there are two perversities, $\underline{p} = (0, 0)$ and $\underline{q} = (0, 1)$.

We can imagine the suspension as the following figure where the front is glued to the back and the left to the right:



The singular set consists of the top and bottom point in the diagram.

In the case of $\underline{p}$, we allow only i-chains η such that $\dim(|\eta| \cap \Sigma) \leq i - 3$ and $\dim(|\delta\eta| \cap \Sigma) \leq i - 4$. Now, we discard any 0-chain intersecting Σ, and for any two points there is a simplex in $X - \Sigma$ connecting the two. Thus $I^{\underline{p}}H_0(X) = \mathbb{C}$. Similarly, $I^{\underline{p}}H_3(X) = \mathbb{C}$, since the generator of $H_3(X)$ is allowable. We have $I^{\underline{p}}H_1(X) = \mathbb{C} \oplus \mathbb{C}$: the bold red and blue 1-chains are generators. Although each are the boundary of a 2-chain in X (for instance the red hatched portion has boundary the bold red 1-chain) any two chain which has either the red or blue 1-cycle as boundary must intersect Σ and thus not allowable. Finally $I^{\underline{p}}H_2(X) = 0$. The allowable 2-chains with no boundary are exactly those that do not intersect Σ, but these are all contractible. moreover, the 3-chain that has such a 2-chain as its boundary is allowable, so they are zero in the intersection homology.

If we compute $I^{\underline{q}}H_\bullet(X)$ in a similar fashion we get:

i	$I^{\underline{p}}H_i(X)$	$I^{\underline{q}}H_i(X)$
0	$\mathbb{C}$	$\mathbb{C}$
1	$\mathbb{C} \oplus \mathbb{C}$	0
2	0	$\mathbb{C} \oplus \mathbb{C}$
3	$\mathbb{C}$	$\mathbb{C}$

3.3. **The intersection pairing.** Our goal now is to show that if we have three perversities $\underline{p}$, $\underline{q}$ and $\underline{r}$ such that $\underline{p} + \underline{q} = \underline{r}$, and an n-dimensional stratified space X, then there is a well defined intersection pairing

$$I^{\underline{p}}H_i(X) \times I^{\underline{q}}H_j(X) \to I^{\underline{r}}H_{i+j-n}(X).$$

Definition 3.13. If $\underline{p}, \underline{q}, \underline{r}$ are perversities for a Whitney stratified space X such that $\underline{p} + \underline{q} = \underline{r}$ and if $C \in C_i^{\underline{p}}(X)$ and $D \in C_j^{\underline{q}}(X)$ then C and D are said to be *dimensionally transverse* if $\dim(|C| \cap |D|) \leq i + j - n$ (i.e. their supports are transverse) and

$$\dim(|C| \cap |D| \cap X_\alpha) \leq i + j - n - k + r(k).$$

Note that the second condition is the first of the two conditions which would make the intersection allowable.

Lemma 3.14. [GM80] *For any two cycles $\eta \in I^{\underline{p}}H_i(X)$ and $\nu \in I^{\underline{q}}H_j(X)$, there is a pair of representative chains that are dimensionally transverse.*

Thus we can define the intersection of two cycles as the sum of the simplicies which appear in the intersection of the two transverse chains along with coefficients calculated using the standard formula.

moreover we have

Theorem 3.15 (Generalized Poincaré Duality)**.** [GM80] *If $\underline{p} + \underline{q} = \underline{t}$ and $i + j = n$, then the intersection pairing*

$$I^{\underline{p}}H_i(X) \times I^{\underline{q}}H_j^c(X) \to I^{\underline{t}}H_0^c(X) \to \mathbb{C}$$

is non degenerate. $I^{\underline{p}}H_i^c(X)$ is taken to be the compactly supported version of intersection cohomology.

Note 3.16. So now each complex variety X has a middle intersection homology which we will denote by $IH_\bullet(X)$. moreover, the intersection pairing becomes $IH_i(X) \times IH_j(X) \to \mathbb{C}$ when $i + j = n$.

3.4. **The IC Sheaf.** At this point it is not obvious that $I^{\underline{p}}H_\bullet$ is a topological invariant. In fact, to show this Goresky and MacPherson [GM83] had to translate the definitions of intersection homology into the language of complexes of sheaves.

Now, given any open $U \subset X$, we can restrict Borel-Moore chains from X to U, i.e. we have a map $C_i(X) \to C_i(U)$. Similarly, an open subset U

gains a stratification from one on X, so we can define $I^{\underline{p}}C_i(U)$ and have a restriction map $C_i^{\underline{p}}(X) \to C_i^{\underline{p}}(U)$. Indeed, one can define sheaves

$$\mathcal{D}_X^{-i}(U) = C_i(U)$$

and

$$\mathcal{I}^{\underline{p}}\mathcal{C}_X^{-i}(U) = C_i^{\underline{p}}(U).$$

moreover the boundary map turns this into a complex.

From such a complex one can recover

$$H_i(X) = H^{-i}(\Gamma(\mathcal{D}_X^\bullet))$$

$$H_i^c(X) = H^{-i}(\Gamma_c(\mathcal{D}_X^\bullet))$$

$$I^{\underline{p}}H_i(X) = H^{-i}(\Gamma(\mathcal{I}^{\underline{p}}\mathcal{C}_X^\bullet))$$

$$I^{\underline{p}}H_i^c(X) = H^{-i}(\Gamma_c(\mathcal{I}^{\underline{p}}\mathcal{C}_X^\bullet))$$

Note 3.17. This construct, of taking the homology of the global sections of a complex of sheaves is known as *hyper cohomology*. $\mathcal{I}^{\underline{p}}\mathcal{C}_X^\bullet$ is known as the *intersection cohomology sheaf* of X (with perversity $\underline{p}$). The sheaf $\mathcal{D}_X^\bullet$ is the *dualizing sheaf*.

4. Sheaves

In general we will be working with sheaves of vector spaces. By $Presh(X)$ denote the category of presheaves of vector spaces (over $\mathbb{C}$) on X and by $Sh(X)$ denote the full subcategory of sheaves on X. The left adjoint of the inclusion $\iota : Sh(X) \hookrightarrow Presh(X)$ is the sheafification map $(\bullet)^+ : Presh(X) \to Sh(X)$. In particular we have

$$Hom_{Sh(X)}((\mathcal{G})^+, \mathcal{F}) \cong Hom_{Presh(X)}(\mathcal{G}, \mathcal{F}).$$

Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{F} \in Sh(X)$, then $f_*\mathcal{F} \in Sh(Y)$ is the *direct image* or *push forward* sheaf on Y. It is defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

This map is functorial and left exact.

Example 4.1. Consider $f : X \to x$, then $f_*\mathcal{F} = \Gamma(\mathcal{F})$.

If one has $\mathcal{G} \in Sh(Y)$, then we also define $f^*\mathcal{G} \in Sh(X)$ the *inverse image* or *pullback* of $\mathcal{G}$ to X. Recall that to do this, one first defines the presheaf

$$(f^{-1}\mathcal{G})(U) = \varinjlim_{f(U) \subset V} \mathcal{G}(V).$$

Then $f^*\mathcal{G} = (f^{-1}\mathcal{G})^+$ and f^* is a functor as well and is right exact.

Example 4.2. If f is the inclusion of a point x into X, then $f^*\mathcal{F} = \mathcal{F}_x$, the stalk of $\mathcal{F}$ at x.

Theorem 4.3. *$f : X \to Y$ be a continuous map, $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$, then*

$$Hom_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Proof. First, we note that

$$Hom_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{Presh(X)}(f^{-1}\mathcal{G}, \mathcal{F}).$$

Suppose that we start with a map $h : Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F})$, then for each $W \subset Y$, we have

$$h_W : \mathcal{G}(W) \to \mathcal{F}(f^{-1}(W)).$$

For each $U \subset X$, we want to construct a map

$$\hat{h}_U : \varinjlim_{f(U) \subset V} \mathcal{G}(V) \to \mathcal{F}(U).$$

Now for each W such that $f(U) \subset W$, the direct limit gives us a map

$$\phi_W : \mathcal{G}(W) \to \varinjlim_{f(U) \subset V} \mathcal{G}(V).$$

moreover, since $f(U) \subset W$, we have $U \subset f^{-1}(W)$ and thus we have a natural restriction $\hat{\phi}_W : \mathcal{F}(f^{-1}(W)) \to \mathcal{F}(U)$. The universal property of direct limits then gives a map $\hat{h}$ as desired, making the following diagram commute:

$$\begin{tikzcd}
\mathcal{G}(W) \arrow[r, "h_W"] \arrow[d, "\phi_W"] & \mathcal{F}(f^{-1}(W)) \arrow[d, "\hat{\phi}_W"] \\
\varinjlim_{f(U) \subset V} \mathcal{G}(V) \arrow[r, "\hat{h}_U"] & \mathcal{F}(U)
\end{tikzcd}$$

We leave the other direction to the reader. □

It will also be important to have the existence of the internal Hom, a sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in Sh(X)$, for sheaves $\mathcal{F} \in Sh(X)$. Its set of sections over an open set $U \subset X$ is $Hom_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$.

moreover, the tensor product of two sheaves is the sheafification of the tensor product of their sections. Then we have the following adjointness results:

Theorem 4.4.

$$f_*\mathcal{H}om(f^*\mathcal{G}, \mathcal{F}) \cong \mathcal{H}om(\mathcal{G}, \mathcal{F})$$

$$Hom_{Sh(X)}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong Hom_{Sh(X)}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{F}))$$

and

$$\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{F})).$$

4.1. **Locally constant and constructible sheaves.** Recall that a constant sheaf $V_X \in Sh(X)$ is the sheafification of the presheaf which takes the same vector space V as its sections over all open sets in X. Its sections over U are simply the locally constant functions $f : U \to V$.

Definition 4.5. $\mathcal{F} \in Sh(X)$ is a *locally constant sheaf* if for each $x \in X$, there exists a neighbourhood U such that $\mathcal{F}|_U$ is constant. If the stalks of $\mathcal{F}$ are also finite dimensional, then we call $\mathcal{F}$ a *local system*.

Example 4.6. Any constant sheaf is thus locally constant. If we consider the punctured plane $X = \mathbb{C}^\times$ and the map $f(z) = z^2$, the direct image of the constant sheaf $\mathbb{C}_X$ on X: $f_*\mathbb{C}_X = \mathbb{C}_X \oplus Q$ where Q is a locally constant sheaf. In particular, the set sections of Q on any annulus containing the origin is 0.

Theorem 4.7. *For a path-connected, locally path connected space, locally simply connected X, there is a bijection between*

$$\left\{ \begin{array}{c} \textit{local systems on } X \\ \textit{up to isomorphism} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{finite-dimensional representations of} \\ \pi_1(X, x_0) \textit{ up to isomorphism} \end{array} \right\}.$$

Proof. Given a locally constant sheaf $\mathcal{F}$ on X, we want to construct a representation of $\pi_1(X, x_0)$. The main idea is to use the fact that if $U \subset X$ such that $\mathcal{F}|_U$ is constant, then the stalks $\mathcal{F}_x$ for $x \in U$ are all naturally isomorphic. Then, given a curve representing an element in $\pi_1(X, x_0)$, we cover the curve in neighbourhoods where $\mathcal{F}$ is the constant sheaf and use compactness to pick finitely many such sets. Chaining the isomorphisms of stalks we obtain an automorphism of $\mathcal{F}_{x_0}$. All one has to do it prove that this is well defined and moreover that it is an group morphism from $\pi_1(X, x_0)$ to $GL(\mathcal{F}_{x_0})$.

The space X has a universal cover $p : \bar{X} \to X$. Given a representation $\tau : \pi_1(X, x_0) \to GL(V)$ for some vector space V, for $U \subset X$, set

$$\mathcal{G}(U) = \{\text{locally constant, } \pi_1(X, x_0) - \text{equivariant maps} \phi : p^{-1}(U) \to V\}.$$

Then $\mathcal{G}$ is a sheaf on X, moreover $\mathcal{G}_{x_0} \cong V$ and one can check that the monodromy action on the stalk is given by τ. ☐

In fact the above bijection can be strengthened to a equivalence of categories: a morphism between two locally constant sheaves will induce a morphism on stalks at x_0 equivariant with respect to the $\pi(X, x_0)$ actions.

Definition 4.8. A sheaf $\mathcal{F}$ on a stratified space X is said to be *constructible* if $\mathcal{F}|_{X_\alpha}$ is locally constant on each stratum X_α.

Example 4.9.
1. The sheaves $\mathcal{D}_X^i$ and $\mathcal{I}^{\underline{p}}\mathcal{C}^i$ defined in section 3.4 are constructible.
2. Suppose that an algebraic group G acts on an variety X with finitely many orbits. Then X stratified by the orbits and this stratification is Whitney: those points in a strata which violate condition B form a subvariety of dimension strictly less than the strata they lie in

[K05], but this condition is G-invariant. In this situation being constructible is the same as being G-equivariant.

4.2. **Direct image with proper support.**

Definition 4.10. Given $\mathcal{F} \in Sh(X)$ and $U \subset X$ open, given $x \in U$, there is map $\mathcal{F}(U) \to \mathcal{F}_x$. For $s \in \mathcal{F}(U)$, we denote s_x is image in $\mathcal{F}_x$, otherwise known as the *germ* of the section over x. The *support* $\mathrm{supp}(s)$ of a section $s \in \mathcal{F}(U)$ is

$$\mathrm{supp}(s) = \{x \in U | s_x \neq 0\}.$$

Example 4.11.

(1) Consider the sheaf $\mathcal{O}_X$ of functions on X. Then if $f \in \mathcal{O}_X(U)$, the support of f as a section is the support of f as a function on U.
(2) Consider a global section in the sheaf of Borel-Moore i-chains $\mathcal{C}_i^{BM}(X)$ on X. The support as a section is the support as an i-chain.

Definition 4.12. Given $f : X \to Y$ a continuous map, there is a map $f_! : Sh(X) \to Sh(Y)$ called the *direct image with proper support*. For $\mathcal{F} \in Sh(X)$, $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$:

$$(f_!\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U)) | f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U \text{ is proper}\}.$$

Lemma 4.13. *$f_!\mathcal{F}$ is a sheaf and $f_!$ is a left exact functor.*

Note 4.14. Given an exact sequence $0 \to A \to B \to C \to 0$ in an abelian category, a functor F from that category is said to be exact if its image is an exact sequence in the image category. This will force the functor to preserver exact sequences of any size. If only $0 \to F(A) \to F(B) \to F(C)$ is exact (i.e. exactness fails at the third term if we include $F(C) \to 0$), then F is left exact. Similarly if $F(A) \to F(B) \to F(C) \to 0$ is exact then F is right exact.

A right adjoint functor is always left exact and a left adjoint functor is always right exact.

In general we will want to find a right adjoint functor to $f_!$ which we would call $f^!$. Unfortunately this does not exist in general. If it did then $f_!$ would be exact. In the case of $f : X \to \{x\}$ with X compact, $f_! = f_*$ and this would say that the global sections functor on a compact space is also exact. But if this where to be true then X would have trivial homology and cohomology since (as we will see) the homology of a space is governed by the failure of Γ to be exact.

Example 4.15.

(1) If f is a proper map, then $f_* = f_!$, since the restriction of f will still be proper.
(2) Consider $f : X \to \{x\}$, then we define $\Gamma_c(\mathcal{F}) = f_!\mathcal{F}$ and call these the global sections of $\mathcal{F}$ with compact support. Consider the sheaf $\mathcal{C}_i^{BM}(X)$ of Borel-Moore i-chains. We know that $\Gamma(\mathcal{C}_i^{BM}(X)) = C_i^{BM}(X)$ and claim that $\Gamma_c(\mathcal{C}_i^{BM}(X)) = C_i(X)$, the compactly supported chains.

5. Derived Functors

Definition 5.1. Given an abelian category $\mathcal{A}$, an object Q is called *injective* if for any $f : A \to Q$ and monomorphism $h : A \to B$, there exists a $g : B \to Q$ with $gh = f$.

An object P is called *projective* if for any $f : P \to B$ and any epimorphism $h : A \to B$, there exist a $g : P \to A$ with $hg = f$.

Lemma 5.2. *An object Q is injective iff* $\hom(\bullet, Q)$ *is exact. An object P is projective iff* $\hom(P, \bullet)$ *is exact.*

Example 5.3. In the category of modules over a ring, a projective module is a free module.

Definition 5.4. An abelian category is said to have enough injectives if for any object A there exists an injective object I and a monomorphism $f : A \to I$. Similarly, it is said to have enough projectives if for any object A there exists a projective object P and an epimorphism $f : P \to A$.

If a category $\mathcal{A}$ has enough injectives, for any object A we can construct its injective resolution:

$$0 \to A \to I^0 \to I^1 \to \cdots ,$$

where each morphism is a monomorphism and each object I^i is injective. Note that this sequence is exact. Something similar can be done for categories with enough projectives.

Definition 5.5. Given a left exact functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ has enough injectives, its i-th right derived functor $R^iF : \mathcal{C} \to \mathcal{D}$ applied to $A \in \mathcal{C}$ is the i-th cohomology of the complex $0 \to F(I^0) \to F(I^1) \to \cdots$ where $0 \to A \to I^0 \to I^1 \to \cdots$ is an injective resolution of A.

Similarly for a right exact functor, we can compute its left derived functor via a projective resolution.

Example 5.6.

(1) The functor $Tor(\bullet, B)$ is the left derived functor of $\bullet \otimes B$. Similarly $Ext(A, \bullet)$ is the right derived functor of the $\hom(A, \bullet)$ functor. In particular, to obtain Tor one takes a free resolution of B and tensors with $\bullet$ and then takes homology.

(2) The category $Sh(X)$ has enough injectives. Since Γ is left exact, it has a right derived functor $R^i\Gamma$. $R^i\Gamma(\mathcal{F})$ which can be seen to be the Čech cohomology $H^i(X, \mathcal{F})$ when X is Hausdorff and paracompact.

6. Categories of Complexes and Derived Categories

Our goal is to produce a better language to talk about derived functors by thinking about categories of complexes of objects and inverting any morphism between two complexes which induce an isomorphism on cohomology.

Let $\mathcal{A}$ be an abelian category. Then $C(\mathcal{A})$ is the category of complexes in $\mathcal{A}$:

- Its objects are (bi-infinite) complexes $\cdots \to A^{-1} \to A^0 \to A^1 \to \cdots$ where $A^i \in \mathcal{A}$ and the maps are typically denoted $d^i : A^i \to A^{i+1}$.
- Its morphisms, $f : A^\bullet \to B^\bullet$ are families $f^i : A^i \to B^i$ such that $f^{i+1}d_A^{i+1} = d_B^i f_i$.

Note 6.1. We often consider a collection of associated categories, $C^+(\mathcal{A})$ of complexes bounded below (i.e. they are 0 eventually on the left), $C^-(\mathcal{A})$ of complexes bounded above and $C^b(\mathcal{A})$ of bounded complexes. In general results about $C(\mathcal{A})$ will be true for each of these other categories as well.

For any complex $A^\bullet \in C(\mathcal{A})$, we can compute its cohomology: Let I be the image object of d^{i-1} and K the kernel of d^i, then the map $I \to A^i$ factors through $K \to A^i$, so we have a map $I \to K$ then $H^i(A^\bullet)$ is the cokernel of this map.

In the category $C(Sh(X))$ this is just the cohomology sheaf of the complex, the quotient of the kernel sheaf by the image sheaf. Recall here that the naive image of the sheaf is just a presheaf and thus the image sheaf is its sheafification.

On the category of complexes we have two natural operations, the first is the shift functor: for a complex $A^\bullet \in C(\mathcal{A})$, we define $(A[n])^i = A^{i+n}$ and the differential $d^i_{A[n]^\bullet} = (-1)^n d^{i_n}_{A^\bullet}$.

For a morphism $f \in Hom_{C(\mathcal{A})}(A^\bullet, B^\bullet)$, we define a new complex, $cone(f) \in C(\mathcal{A})$ by $(cone(f))^i = A^{i+1} \oplus B^i$ with differential

$$d^i_{cone(f)} = \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix}.$$

This is called the *mapping cone* of f.

Definition 6.2. A complex $A^\bullet$ is said to be *acyclic* if $H^i(A^\bullet) = 0$ for all i.

Lemma 6.3. *There exists a short exact sequence*

$$0 \to B^\bullet \to cone(f) \to A^\bullet[1] \to 0.$$

Moreover this gives a long exact sequence in homology:

$$\cdots \to H^{n-1}(cone(f)) \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(cone(f)) \to \cdots .$$

In particular $cone(f)$ is acyclic *iff f induces an isomorphism on homology.*

Definition 6.4. A morphism $f : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if $H^i(f)$ is an isomorphism.

Definition 6.5. A homotopy between two maps $f, g \in Hom_{C(\mathcal{A})}(A^\bullet, B^\bullet)$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that

$$d_B^{i-1}h^i + h^{i+1}d_A^i = f - g.$$

If such an h exists then f and g are said to be *homotopic*, $f\ g$.

Definition 6.6. The *homotopy category* over $\mathcal{A}$, denoted $K(\mathcal{A})$ has the same objects as $C(\mathcal{A})$ but we set

$$Hom_{K(\mathcal{A})}(A^\bullet, B^\bullet) = Hom_{C(\mathcal{A})}(A^\bullet, B^\bullet)/\{f : A^\bullet \to B^\bullet | f \sim 0\}.$$

The overall picture here is that we only care about the cohomology of complexes and when f and g are homotopic, they induce the same map on cohomology. Thus taking the i-th cohomology of a complex descends to a functor $H^i : K(\mathcal{A}) \to \mathcal{A}$.

Unfortunately by performing this operation, the resulting category is no longer abelian and is only additive. This means, in particular that one can no longer speak of short exact sequences in the category. We can how ever discuss a structure on the $K(\mathcal{A})$ is as close as we can get to an abelian structure: the structure of a *triangulated category.*

It should be noted that the category $\mathcal{A}$ is a full subcategory of $K(\mathcal{A})$, via the complexes concentrated in degree zero.

6.1. **Triangulated Categories.** The definition of a triangulated category is due essentially to Verdier in PhD thesis under Grothendieck.

A *triangulated category* is an additive category $\mathcal{A}$ with two extra structures. The first is an autoequivalence $\Sigma : \mathcal{A} \to \mathcal{A}$ called *suspension.* This is often known as the *translation* functor, in which case the usual notation is $\Sigma^n A = A[n]$. The second is a class of triangles,

$$X \to Y \to Z \to X[1],$$

with the usual condition that the composition of any two adjacent arrows is 0, called *distinguished triangles.*

The collection of distinguished triangles must satisfy the following four axioms:

TR 1 The triangle

$$X \xrightarrow{\text{id}} X \to 0 \to X[1]$$

is distinguished. Moreover, for each $f : X \to Y$, we have distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

Z is then known as a mapping cone for f. The class is closed under isomorphism.

TR 2 The rotation of a distinguished triangle is distinguished. That is, given

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

the triangles

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z$$

are distinguished.

TR 3 Given maps f and g which make the square on the left commute, the morphism h exists (not necessarily uniquely):

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

It should be noted that this axiom will also follow from the next axiom.

TR 4 This is known as the octahedral axiom. Starting with 3 distinguished triangles,

$$X \xrightarrow{u} Y \to Z' \to X[1]$$
$$Y \xrightarrow{v} Z \to X' \to Y[1]$$
$$X \xrightarrow{vu} Z \to Y' \to X[1],$$

we have a fourth distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that everything in the following diagram commutes:

$$\begin{tikzcd}
& Y' \arrow[dr, dashed] \arrow[ddl] & \\
Z' \arrow[ur, dashed] \arrow[d] & & X' \arrow[ll, dashed] \arrow[ddl] \\
X \arrow[rr] \arrow[dr] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[ur] \arrow[uul] &
\end{tikzcd}$$

The idea behind the axioms is that is that although given a map $f : X \to Y$ in the category, we do not have a kernel an cokernel, we will have a *homotopy kernel* and *homotopy cokernel*. Since a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1]$$

exists, we consider $Z[-1] \to X$ to be homotopy kernel and $Y \to Z$ to be the homotopy cokernel.

With this thinking, the first three axioms can be thought of as saying that:

- The identity map has zero homotopy kernel and cokernel.
- Every map has a homotopy kernel and cokernel.

- Any map is the homotopy kernel of its homotopy cokernel and every map is the homotopy cokernel of its kernel.
- Homotopy kernels and cokernels are 'functorial'

Lemma 6.7. *If $f : A \to B$ is an isomorphism, then*

$$A \xrightarrow{f} B \to 0 \to A[1]$$

is distinguished.

Proof. All the vertical arrows in the following diagram are isomorphisms and all squares commute:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & A[1]
\end{tikzcd}$$

Thus the two triangles are isomorphic and since the first is distinguished so is the second. $\square$

Lemma 6.8. *Given a morphism $f : X \to Y$, for any two distinguished triangles $X \xrightarrow{f} Y \to Z \to X[1]$ and $X \xrightarrow{f} Z' \to X[1]$, there is a (non-unique) isomorphism between Z and Z'.*

Proof. Given a distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$, and an object U, the sequence of abelian groups

$$Hom(U, A) \to Hom(U, B) \to Hom(U, C)$$

is exact. Indeed, given $f : U \to B$ such that $v \circ f = 0$ (i.e. f is in the kernel), then consider

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & U[1] \arrow[r, "-\mathrm{id}_U[1]"] \arrow[d, dashed, "h"] & U[1] \arrow[d, "f[1]"] \\
B \arrow[r, "v"] & C \arrow[r, "w"] & A[1] \arrow[r, "-u[1]"] & B[1]
\end{tikzcd}$$

Since h exists via axiom 3, we see that $f = u \circ h$ and thus is in the image.

Then it follows that

$$\cdots \to Hom(U, A) \to Hom(U, B) \to Hom(U, C) \to Hom(U, A[1]) \to \cdots$$

is exact for any distinguished triangle.

Applying this with $U = Z'$ to

$$\begin{tikzcd}
\cdots \arrow[r] & X \arrow[r] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[r] \arrow[d, "\mathrm{id}_X[1]"] & Y[1] \arrow[r] \arrow[d, "\mathrm{id}_Y[1]"] & \cdots \\
\cdots \arrow[r] & X \arrow[r] & Y \arrow[r] & Z' \arrow[r] & X[1] \arrow[r] & Y[1] \arrow[r] & \cdots
\end{tikzcd}$$

and using the 5-lemma shows that the induced map $h : Hom(Z', Z) \to Hom(Z', Z')$ is an isomorphism of groups. Thus there exists a $g : Z' \to Z$ such $h \circ g = \mathrm{id}_{Z'}$. What about $g \circ h$? Well, the induced map $h : Hom(Z, Z) \to Hom(Z, Z')$ is also isomorphism and the image of $g \circ h$ is $h \circ g \circ h = h$, but the image of id_Z is also h, so $g \circ h = \mathrm{id}_Z$. $\square$

Definition 6.9. A functor $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$, $\mathcal{B}$ are triangulated categories is called a *triangulated functor* if $F\Sigma$ is naturally isomorphic to ΣF and sends distinguished triangles to distinguished triangles.

Example 6.10. The category $K(\mathcal{A})$ is triangulated for any abelian category $\mathcal{A}$ with distinguished triangles generated by $X \xrightarrow{f} Y \to cone(f) \to X[1]$.

Indeed, this now says that if $f : X \to Y$ then $cone(f) \cong 0$ in $K(\mathcal{A})$. Moreover, the sequence

$$\cdots \to H^i(X^\bullet) \to H^i(Y^\bullet) \to H^i(Z^\bullet) \to H^{i+1}(X^\bullet) \to \cdots$$

is exact.

Lemma 6.11. *Given a distinguished triangle* $X^\bullet \to Y^\bullet \to Z^\bullet \to Z^\bullet[1]$ *in* $K(\mathcal{A})$ *the sequence*

$$H^0(X^\bullet) \to H^0(Y^\bullet) \to H^0(Z^\bullet)$$

is exact.

Proof. Let $f : X \to Y$ be the morphism in the triangle and a be a representative in $C(\mathcal{A})$ then we have an isomorphism of complexes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_X[1]"] \\
X \arrow[r, "f"] & Y \arrow[r] & C_a \arrow[r] & X[1]
\end{tikzcd}$$

Applying H^0 leaves the vertical arrows as isomorphisms and makes the bottom row exact. This forces the top row to be exact as well. $\square$

6.2. **The Derived Category.** Our goal now is to define a category whose objects are complexes and morphisms are morphisms of complexes but with the added condition that all quasi-isomorphisms are invertible.

One way of doing this is through a universal property: Given a collection S of morphisms in a category $\mathcal{A}$, we could consider a category $\mathcal{A}[S^{-1}]$, i.e. $\mathcal{A}$ localized at S, equipped with a functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ such that $Q(s)$ for $s \in S$ is invertible. This should satisfy the following universal property: $F : \mathcal{A} \to \mathcal{B}$ with $F(s)$ invertible for all $s \in S$, there should exist a unique functor $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $G \circ F = Q$.

This category exists and is unique, but the definition of morphisms is no simple. In the case that we have a category with a triangulated structure and a set of morphisms compatible with this structure, then the definition becomes much simpler. By compatible, we mean that the set S is closed under translation, and in third triangulated category axiom, if the morphisms $f, g \in S$, then $h \in S$.

Definition 6.12. The *derived category* of $\mathcal{A}$, denoted $D(\mathcal{A})$ has the same objects as $C(\mathcal{A})$ but its morphisms are given by equivalence classes of *roofs*. That is, a morphism from an object $A^\bullet$ to an object $B^\bullet$ is a pair $f : X^\bullet \to A^\bullet$, a quasi-isomorphism, and $g : X^\bullet \to B^\bullet$ both morphisms in $K(\mathcal{A})$:

$$\begin{tikzcd} & X^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

Two roofs are equivalent if there is a third roof such that the following diagram commutes:

$$\begin{tikzcd} & Z^\bullet \arrow[dl] \arrow[dr] & \\ X^\bullet \arrow[d] \arrow[drr] & & Y^\bullet \arrow[d] \arrow[dll] \\ A^\bullet & & B^\bullet \end{tikzcd}$$

First, we should check that quasi-isomorphisms are in fact compatible with triangulated structure on $K(\mathcal{A})$. First, it is clear that the shift of a quasi-isomorphism is still a quasi-isomorphism. The only thing to check is what happens when we have:

$$\begin{tikzcd}
X^\bullet \arrow[r] \arrow[d, "f"] & Y^\bullet \arrow[r] \arrow[d, "g"] & Z^\bullet \arrow[r] \arrow[d, "h"] & X^\bullet[1] \arrow[d, "f[1]"] \\
X'^\bullet \arrow[r] & Y'^\bullet \arrow[r] & Z'^\bullet \arrow[r] & X'^\bullet[1]
\end{tikzcd}$$

Where f and g are quasi-isomorphisms. If we apply the cohomology functor then the sequence on top and bottom become exact and the maps $H^*(f)$ and $H^*(g)$ are isomorphisms, so by the 5-lemma, the map $H^*(h)$ is also an isomorphism.

The next issue is that it is not obvious how one composes roofs. Indeed, take

$$\begin{tikzcd}
& Z^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & & Z'^\bullet \arrow[dl, "h"'] \arrow[dr, "i"] & \\
U^\bullet & & V^\bullet & & W^\bullet
\end{tikzcd}$$

With f and h quasi-isomorphisms. How does one use this data to define a roof from $U^\bullet$ to $W^\bullet$? If it is always possible to find k, a quasi-isomorphism and l that make a commuting square the pair g and h:

$$\begin{tikzcd}
& & Z''^\bullet \arrow[dl, "k"'] \arrow[dr, "l"] & & \\
& Z^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & & Z'^\bullet \arrow[dl, "h"'] \arrow[dr, "i"] & \\
U^\bullet & & V^\bullet & & W^\bullet
\end{tikzcd}$$

Then the roof given by $f \circ k$ and $i \circ l$ would work.

Consider a distinguished triangle

$$Z^\bullet \xrightarrow{g} V^\bullet \xrightarrow{\iota} M^\bullet \to Z^\bullet[1]$$

for g. Then consider

$$Z'^\bullet \xrightarrow{\iota\circ h} M^\bullet \to Z''^\bullet[1] \to Z'^\bullet[1]$$

a distinguished triangle for $\iota \circ h$. Then we have

$$\begin{tikzcd}
Z'^\bullet \arrow[r, "\iota \circ h"] \arrow[d, "h"] & M^\bullet \arrow[r] \arrow[d, "\mathrm{id}_M"] & Z''^\bullet \arrow[r, "{l[1]}"] \arrow[d, "{k[1]}"] & Z'^\bullet[1] \arrow[d, "{h[1]}"] \\
V^\bullet \arrow[r, "\iota"] & M^\bullet \arrow[r] & Z^\bullet[1] \arrow[r, "{-g[1]}"] & V^\bullet[1]
\end{tikzcd}$$

In particular the last square commutes and the map $k[1]$ is a quasi-isomorphism. If we shift it back by 1 and move the -1 unto the map $l[1]$, then we see that k and l are as needed.

There is a natural functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$, which is the identity on objects and takes a morphism $f : X^\bullet \to Y^\bullet$ to a roof (id_X, f).

Next, for a quasi-isomorphism $q : X \to Y$, it is now invertible, since we can compose the roof (id_X, q) and (q, id_X) to obtain the roof $(\mathrm{id}_X, \mathrm{id}_X)$ and similarly we can compose them in the opposite direction to obtain (q, q) which one can check is equivalent to the identity.

Theorem 6.13. *Let Q be the set of quasi-isomorphisms in $K(\mathcal{A})$, then the functor $F : K(\mathcal{A})[Q^{-1}] \to D(\mathcal{A})$ is an isomorphism of categories. Moreover, the induced functor $\tilde{F} : C(\mathcal{A})[Q^{-1}] \to D(\mathcal{A})$ is also an isomorphism of categories.*

One non-obvious corollary is that in the category $C(\mathcal{A})[Q^{-1}]$, homotopic maps are equal!

An important question is ask oneself is why we cannot simply use the method of roofs to construct $D(\mathcal{A})$ directly from the category $C(\mathcal{A})$. The key problem is that $C(\mathcal{A})$ is not a triangulated category and therefore we cannot expect to be able to pull back a pair (q, f) where q is a quasi-isomorphism to a pair (g, q') where q' is a quasi-isomorphism.

Example 6.14. Consider an abelian category $\mathcal{A}$ with enough injectives. Then suppose that we have $M, N \in \mathcal{A}$ such that $Ext^i(M, N) \neq 0$ for some $i > 0$. Then take $f \in Ext^i(M, N)$, $f \neq 0$, it can be considered as a morphism between $M \in C(\mathcal{A})$ considered as a complex in concentrated in degree 0 and $I_N[i]$ the injective resolution of N. If $s : N \to I_N$ is the resolution, then s is a quasi-isomorphism, then suppose that M', g and t exist, making this square commute with t a quasi-isomorphism:

$$\begin{tikzcd}
M' \arrow[r, "t"] \arrow[d, "g"] & M \arrow[d, "f"] \\
N[i] \arrow[r, "{s[i]}"] & I_N[i]
\end{tikzcd}$$

Then, since t is a quasi-isomorphism and $H^0(M) = M$ (considered as a complex in degree 0), it follows that t^0 is nonzero, since it must induce an isomorphism to $H^0(M')$. Note that $f^i = 0$ for all $i \neq 0$, thus $f \circ t$ is concentrated in degree 0 and $f \circ t \neq 0$

On the other hand the map $s[i]$ is concentrated in degree $-i$, so $s[i] \circ g$ is also concentrated in degree $-i$. But this means that $s[i] \circ g \neq f \circ t$ and the diagram cannot commute.

The category $D(\mathcal{A})$ is still triangulated, with the same shift functor, but the class of distinguished triangles is now somewhat larger: the image of any distinguished triangle of $K(\mathcal{A})$ under $Q : K(\mathcal{A}) \to D(\mathcal{A})$ is considered to be distinguished and thus any triangle isomorphic is distinguished as well. I.e. triangles in $K(\mathcal{A})$ that are quasi-isomorphic to distinguished triangle in $K(\mathcal{A})$ are now distinguished in $D(\mathcal{A})$.

Lemma 6.15. *The functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ is triangulated.*

7. Derived Functors

The main question we wish to answer now, is when a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories gives rise to a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$. An additive functor will always at least give $F : K(\mathcal{A}) \to K(\mathcal{B})$, since it must preserve homotopies. Moreover, when F is exact and $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then since $cone(f)$ is exact, so is $F(cone(f)) = cone(F(f))$, so $F(f)$ is exact as well. Thus an exact functor preserves quasi-isomorphisms. Moreover one can check that it must also preserve distinguished triangles.

In the case of left or right exact functor $F : \mathcal{A} \to \mathcal{B}$, it turns out that we can lift them to $D^+(\mathcal{A}) \to D^+(\mathcal{B})$ or $D^-(\mathcal{A}) \to D^-(\mathcal{B})$ respectively if $\mathcal{A}$ has enough injectives or projectives.

We will concentrate on the case of left exact functors since they will be of most use to us.

Lemma 7.1. *Given a short exact sequence $0 \to A \to B \to C \to 0$ in an abelian category $\mathcal{A}$ that has enough injectives, there exists a short exact sequence $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$ of injective resolutions.*

Proof. Let
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$
and set $I^\bullet$ and $K^\bullet$ to each be an injective resolution of A and C respectively. Let $u : A \to I^0$ and $v : C \to K^0$ be the corresponding monomorphisms. Then set $J^i = I^i \oplus K^i$. Not that we do not take $J^\bullet = I^\bullet \oplus K^\bullet$. In particular, we begin by defining the map $w : B \to I^0 \oplus K^0$ by $w = \alpha \oplus vg$ where $\alpha : B \to I^0$ as given by the universal property of I^0 so that $\alpha f = u$. This is a monomorphism, since its kernel is a subobject of $\ker vg = \ker v = A$ via $f : A \to B$. Thus $\ker w = \ker w \circ f = \ker u = 0$.

Consider $\tilde{B} = \mathrm{coker} w = (I^0 \oplus K^0)/\mathrm{im}(w)$, now the projection onto the second fact of $I^0 \to K^0$ sends $\mathrm{im}(w) = (\alpha \oplus (v \circ g))(B)$ to $v \circ g(B) = v(C)$ since g is surjective. I.e. the projection map lifts to $\tilde{B}$ and it image is $K^0/v(C) = \mathrm{coker} v$.

Similarly if we now take the inverse of the kernel of this map under the first projection we get $I^0/u(A) = \mathrm{coker}u$. Thus the we have a short exact sequence of coker and the $\mathrm{coker}u$ and $\mathrm{coker}v$ are equipped with monomorphisms to I^1 and K^1 and we begin the argument again from the top. $\square$

Theorem 7.2. *Any complex $A^\bullet \in C^+(\mathcal{A})$ has a resolution by injectives. That is, there exists a complex $I^\bullet \in C^+(\mathcal{A})$ and a quasi-isomorphism $t : A^\bullet \to I^\bullet$.*

Proof. For convenience we will assume that $A^i = 0$ for $i < 0$:

$$0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \to \cdots .$$

Next using the above lemma, we generate a short exact sequence

$$0 \to R^{\bullet,i} \to S^{\bullet,i} \to T^{\bullet,i} \to 0$$

of injective resolutions for

$$0 \to \mathrm{im}(d^{i-1}) \to \ker(d^i) \to H^i(A) \to 0.$$

Next we apply the lemma again to obtain a short exact sequence

$$0 \to S^{\bullet,i} \to I^{\bullet,i} \to R^{\bullet,i+1} \to 0,$$

corresponding to

$$0 \to \ker(d^i) \to A^i \to \mathrm{im}(d^i) \to 0.$$

This gives rise to a bicomplex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^{1,0} \arrow[u, "d_v^{1,0}"] \arrow[r, "d_h^{1,0}"] & I^{1,1} \arrow[u, "d_v^{1,1}"] \arrow[r, "d_h^{1,1}"] & I^{1,2} \arrow[u, "d_v^{1,2}"] \arrow[r, "d_h^{1,2}"] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[u, "d_v^{0,0}"] \arrow[r, "d_h^{0,0}"] & I^{0,1} \arrow[u, "d_v^{0,1}"] \arrow[r, "d_h^{0,1}"] & I^{0,2} \arrow[u, "d_v^{0,2}"] \arrow[r, "d_h^{0,2}"] & \cdots \\
0 \arrow[r] & A^0 \arrow[u, "\iota_0"] \arrow[r, "d^0"] & A^1 \arrow[u, "\iota_1"] \arrow[r, "d^1"] & A^2 \arrow[u, "\iota_2"] \arrow[r, "d^2"] & \cdots
\end{tikzcd}$$

The $d_v^{i,h}$ are simply the corresponding morphisms from the injective resolutions. The $d_h^{i,j}$ are the compositions $I^{i,j} \to R^{i,j+1} \to S^{i,j+1} \to I^{i,j+1}$. In particular the composition $d_h^{i,j+1} \circ d_h^{i,j} = 0$ since it factors through the exact sequence $S^{i,j+1} \to I^{i,j+1} \to R^{i,j+2}$.

This complex has several special features, the main of which is that the horizontal cohomology along a single vertical is an injective resolution of the cohomology of the complex $A^\bullet$.

The total complex of $I^{\bullet,\bullet}$ has i-th object given by $\bigoplus_{j+k=i} I^{j,k}$ and differential given by an appropriate sum of $d_v^{j,k}$ and $(-1)^j d_h^{j,k}$. Since all squares commute and all horizontal or vertical compositions are 0, this gives differential for the complex. Moreover as the objects are finite direct sums of injectives, they are injective themselves.

Finally, the complex comes equipped with an injective map from $A^\bullet$ via $\iota^\bullet$. Using spectral theory, since the columns of $I^{\bullet,\bullet}$ are exact everywhere but the 0 row, the spectral sequence degenerates at the E_2 page. Thus the cohomology of the total complex is isomorphic to the cohomology of $A^\bullet$ and the monomorphism $\iota^\bullet$ must induce such an isomorphism. □

We are now in a position to define the right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ associated to a left exact functor $F : \mathcal{A} \to \mathcal{B}$:

Definition 7.3. Given a complex $A^\bullet \in D^+(\mathcal{A})$, then $A^\bullet$ is isomorphic to any injective resolution $I^\bullet$. Let $RF(A^\bullet) = F(I^\bullet)$.

Of course, for this to be well defined there is something to prove, that $F(I^\bullet)$ is independent of the choice $I^\bullet$. But one can check that for any other injective resolution, one automatically gets a quasi-isomorphism between it and the one we constructed. Left exact functors preserve quasi-isomorphisms between injective complexes because they preserve exact injective complexes.

It should be noted that $H^i(RF(A^\bullet)) = R^iF(A^\bullet)$.

7.1. **Derived functors and Sheaves.** If X is a space as before, by $D(X)$ we mean $D(Sh(X))$ and similarly by $D^b(X) = D^b(Sh(X))$. Given $f : X \to Y$, f continuous, we have 3 natural functors $f_* : Sh(X) \to Sh(Y)$, $f^* : Sh(Y) \to Sh(X)$ and $f_! : Sh(X) \to Sh(Y)$. Recall that f^* is exact and f_* and $f_!$ are left exact, so f^* gives a functor $f^* : D^b(Y) \to D^b(X)$ and others gives right derived functors $Rf_* : D^b(X) \to D^b(Y)$ and $Rf_! : D^b(X) \to D^b(Y)$.

Naturally, we will also have right derived global sections $R\Gamma$ and $R\Gamma_c$ with compact support.

Definition 7.4. Given $A \in D^b(X)$, its *hypercohomology* is $\mathbb{H}^i(X, A^\bullet) = H^i(R\Gamma(A^\bullet))$. Analogously, hypercohomology with compact supports is $\mathbb{H}^i_c(X, A^\bullet) = H^i(R\Gamma_c(A^\bullet))$.

Then if $A \in Sh(X)$, $\mathbb{H}^i(X, A^\bullet) = H^i(X, A)$ i.e. hypercohomology of a sheaf concentrated in degree 0 is sheaf cohomology. In particular if X is locally contractible, $H^i(X, \mathbb{C}) = H^i(X, \mathbb{C}_X) = \mathbb{H}^i(X, \mathbb{C}^\bullet_X)$.

Lemma 7.5. *$f : X \to Y$ continuous,*

$$\mathbb{H}^\bullet(X, A^\bullet) \cong \mathbb{H}^\bullet(Y, Rf_*A^\bullet).$$

Proof. One checks that $R\Gamma(X, A^\bullet) \cong R\Gamma(Y, Rf_*A^\bullet)$. This is due to the fact that f^* is exact. The adjunction between f^* and f_* then shows that f_* preserves injectives. □

If we take $A^\bullet, B^\bullet \in D^b(X)$, then $A^\bullet \otimes B^\bullet$ is the total complex of the double complex $A^p \otimes B^q$. The functor $A^\bullet \overset{L}{\otimes} \bullet$ has a left derived version, obtained by taking a flat resolution of the input. Moreover one has

$$A^\bullet \overset{L}{\otimes} B^\bullet \cong B^\bullet \overset{L}{\otimes} A^\bullet.$$

There is also a functor $\mathcal{H}om^\bullet(B^\bullet, \bullet) : C^b(Sh(X)) \to C^b(Sh(X))$. It has

$$(\mathcal{H}om(B^\bullet, A^\bullet))^n = \prod_p \mathcal{H}om(B^p, A^{n+p}).$$

The differential can be considered to be a collection of maps $d_{jk,mn} : \mathcal{H}om(A^j, B^k) \to \mathcal{H}om(A^m, B^n)$ given by

$$d_{jk,mn} = \begin{cases} f \mapsto d_B^k \circ f & \text{if } m = j \text{ and } n = k+1, \\ f \mapsto (-j)^j f \circ d_A^{j-1} & \text{if } m = j-1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly one can define a non-sheaf version, $Hom^\bullet(A^\bullet, \bullet)$ which gives a complex of $\mathbb{C}$ vector spaces. As usual, one can check that $Hom^\bullet$ is just the global sections of $\mathcal{H}om^\bullet$. Both functors are left exact, so enjoy a right derived version $R\mathcal{H}om^\bullet$ and $RHom^\bullet$. For consistency sake, it should be noted that the right derived version of $\mathcal{H}om^\bullet(\bullet, A^\bullet)$ is naturally isomorphic to $R\mathcal{H}om^\bullet$.

Lemma 7.6.

$$RHom^\bullet(B^\bullet, A^\bullet) \cong R\Gamma(X, R\mathcal{H}om^\bullet(B^\bullet, A^\bullet)).$$

Lemma 7.7.

$$H^0(RHom^\bullet(B^\bullet, A^\bullet)) \cong Hom_{D^b(X)}(B^\bullet, A^\bullet)$$

.

Lemma 7.8.

$$R\mathcal{H}om^\bullet(A^\bullet \overset{L}{\otimes} B^\bullet, C^\bullet) \cong R\mathcal{H}om^\bullet(A^\bullet, R\mathcal{H}om^\bullet(B^\bullet, C^\bullet)).$$

The above three lemmas can be combined to tell us that the $\overset{L}{\otimes}$ and $R\mathcal{H}om^\bullet$ are an adjoint pair of functors over $D^b(X)$.

Moreover we can define the $\mathcal{E}xt^i(B^\bullet, A^\bullet) = H^i(R\mathcal{H}om^\bullet(B^\bullet, A^\bullet))$. Then

$$Ext^i(B^\bullet, A^\bullet) = H^i(RHom^\bullet(B^\bullet, A^\bullet)) = H^0(RHom^\bullet(B^\bullet, A^\bullet[i])) \cong Hom_{D^b(X)}(B^\bullet, A^\bullet[j]).$$

The pair (f^*, Rf_*) is a pair of adjoint functors over the derived category. Indeed we have

Lemma 7.9. *If $f : X \to Y$ is continuous, $A^\bullet \in D^b(X)$ and $B^\bullet \in D^b(Y)$ we have:*

$$R\mathcal{H}om^\bullet(B^\bullet, Rf_* A^\bullet) \cong Rf_* R\mathcal{H}om^\bullet(f^* B^\bullet, A^\bullet).$$

Our goal is to define a right adjoint $f^!$ to the functor $Rf_!$. In general this functor between derived categories will not be a derived functor of a functor between the underlying categories of sheaves.

Definition 7.10. Given $f : X \to Y$, continuous and $A^\bullet \in D^b(X)$, begin by defining presheaf

$$\hat{f}^! A^\bullet(U) = Hom^\bullet(f_! K^\bullet, I^\bullet)$$

where $I^\bullet$ is an injective resolution of $A^\bullet$ and $K^\bullet$ is an injective resolution of $\mathbb{C}^\bullet_U$. $f^! A^\bullet$ is then the sheaf associated to $\hat{f}^! A^\bullet$.

Note 7.11. Often there is a more general class of resolutions (i.e. other than injective) that one can use to compute derived functors. In particular in the above definition, if X is locally contractible, $K^\bullet$ can be replaced by sheafification of the complex of presheaves $C^\bullet$ where $C^i(U)$ is the set of i-cochains on U. Let $\mathcal{C}^\bullet_X$ denote this sheafification. This complex is exact everywhere but at 0-th degree. All we have to do is very the exactness of the stalks at each $x \in X$. Since X is locally contractible, there exists a contractible neighbourhood $x \in U$. $\mathcal{C}^\bullet_X|_U = \mathcal{C}^\bullet_U$ and its hypercohomology is cohomology of a point and thus the stalk $(\mathcal{C}^\bullet_X)_x$ is exact everywhere but degree 0 (where the homology is $\mathbb{C}$). This resolution is also soft, so we do not need to take an injective resolution before applying a derived functor.

Example 7.12.

- Let us apply the definition in two cases. The first is $f : X \to \{pt\}$ and $A^\bullet = \mathbb{C}^\bullet_X$, i.e. the complex of vector spaces with $\mathbb{C}$ in degree 0. Now,

$$\begin{aligned}(\hat{f}^!(U))^i &= Hom^i(f_! \mathcal{C}^i_U, \mathbb{C}^\bullet)\\ &= \oplus_j Hom_{Vect}(\Gamma_c(\mathcal{C}^j_U), \mathbb{C}^{j+i})\\ &= Hom_{Vect}(\Gamma_c(\mathcal{C}^{-i}_U), \mathbb{C})\\ &= (\Gamma_c(C^{-i}(U)))^*\\ &= C^{BM}_{-i}(U).\end{aligned}$$

  Where $\mathcal{C}^i_U$ is the sheaf of i-cochains on U and $C^{BM}_i(U)$ is the set of Borel-Moore i-chains. This means that $(f^!\mathbb{C}_X)^i$ is sheafification of presheaf whose sections over U are the Borel-Moore $-i$-chains. Its hypercohomolgy is $H^{BM}_i(X)$.
- What about $f : \{x\} \to X$? The complex $f^! A^\bullet$ is often called the *costalk* at x of the complex $A^\bullet$.
- If $f : X \hookrightarrow Y$, where X is a closed subset of Y, then we can define $f^! : Sh(Y) \to Sh(X)$ which is a right adjoint to $f_!$. Indeed, given a sheaf $\mathcal{F}$ on Y we define a sheaf $\mathcal{F}^X$ such that $\mathcal{F}^X(U) = \{s \in \mathcal{F}(U) | \text{supp}(s) \subset X\}$. Then set $f^!\mathcal{F} = f^*\mathcal{F}^X$. Moreover this functor is left exact and preserves injectives. Thus it has a right derived functor which must be right adjoint to $Rf_!$ and thus be $f^!$ as defined above.

If we consider again the map $f : \{x\} \to X$, we have $f^! : Sh(X) \to Vect$. Then given $\mathcal{F}$ on X and $U \subset V$ open in X, there is a natural inclusion $\Gamma_c(\mathcal{F}, U) \to \Gamma_c(\mathcal{F}, V)$, extension by 0. Then the $f^!\mathcal{F} = \lim_{\overleftarrow{x \in U}} \Gamma_c(\mathcal{F}, U)$. Moreover its right derived functor will be $f^! : D^b(Y) \to D^b(X)$.

Definition 7.13. If $f : X \to \{\bullet\}$ then $f^!\mathbb{C} = \mathbb{D}_X$, this is also known as the *dualizing sheaf* on X.

Once the dualizing sheaf was defined, Verdier defined the operation of dualizing a complex of sheaves. This was inspired by, and implements Poincaré duality on the level of complexes.

Definition 7.14. Given a complex $A^\bullet$ on X, the Verdier dual is

$$\mathcal{D}A^\bullet = R\mathcal{H}om^\bullet(A^\bullet, \mathbb{D}_X).$$

This is quite powerful. In particular, the following results are true:

Lemma 7.15. *There is a quasi-isomorphism $A^\bullet \to \mathcal{D}\mathcal{D}A^\bullet$, in particular the dual of the dual of a complex is isomorphic to the original complex in $D^b(X)$. Also $\mathcal{D}(A^\bullet[n]) = (\mathcal{D}A^\bullet)[-n]$.*

Lemma 7.16. *$\mathbb{D}_X$ is constructible with respect to any Whitney stratification and $\mathbb{D}_X = f^!\mathbb{D}_Y$ for any map $f : X \to Y$. If X is smooth, oriented of real dimension m, then $\mathbb{D}_X = \mathbb{C}_X[m]$.*

Lemma 7.17. *If $f : X \to Y$, continuous, then $Rf_! = \mathcal{D} \circ Rf_* \circ \mathcal{D}$. Moreover $f^! = \mathcal{D} \circ f^* \circ \mathcal{D}$ which could be used a definition.*

Example 7.18.

- In the category $D^b(\{x\})$, since the dualizing sheaf is just the vector space $\mathbb{C}$ concentrated in degree zero, the Verdier dual of any complex of vector spaces is itself. Given $f : \{x\} \to X$, since f^* is exact, we have $f^! = f^*\mathcal{D}$. In particular, if X is smooth and oriented, then

$$f^!\mathbb{C}^\bullet_X = \mathcal{D}f^*\mathcal{D}\mathbb{C}^\bullet_X = \mathcal{D}(f^*\mathbb{C}^\bullet_X[n]) = \mathcal{D}(\mathbb{C}^\bullet[n]) = \mathbb{C}^\bullet[-n],$$

where $n = \dim_{\mathbb{R}} X$.
- Note if X is locally contractible we have

$$\begin{aligned}
\mathbb{H}^i(X, \mathbb{C}^\bullet_X) &= H^i(X, \mathbb{C}) \\
\mathbb{H}^i_c(X, \mathbb{C}^\bullet_X) &= H^i_c(X, \mathbb{C}) \\
\mathbb{H}^i(X, \mathbb{D}_X) &= H^{BM}_{-i}(X, \mathbb{C}) \\
\mathbb{H}^i_c(X, \mathbb{D}_X) &= H_{-i}(X, \mathbb{C}))
\end{aligned}$$

Moreover if X is smooth and orientable and even dimensional, if we set $n = \dim_R X/2$, then $\mathcal{D}\mathbb{C}^\bullet_X[n] \cong \mathbb{C}^\bullet_X[n]$. But if apply the above 4 cohomologies, we see that

$$H^{n-i}(X, \mathbb{C}) \cong H^{BM}_{n+i}(X, \mathbb{C})$$

and

$$H_c^{n-i}(X,\mathbb{C}) \cong H_{n+1}(X,\mathbb{C}).$$

8. Perverse Sheaves

The category $D^b(X)$ comes equipped with a standard t-structure. This consists of two full subcategories $D^b(X)^{\leq 0}$ and $D^b(X)\geq 0$ of complexes which are exact at degree $i > 0$ an $i < 0$ respectively. Their intersection $D^b(X)^{\leq 0} \cap D^b(X)^{\geq 0} = Sh(X)$.

By $D^b(X)^{\leq n}$ we mean $D^b(X)^{\leq 0}[-n]$ and by $D^b(X)^{\geq n}$ we mean $D^b(X)^{\geq 0}[-n]$. The two inclusions $\iota^{\leq 0} : D^b(X)^{\leq 0} \to D^b(X)$ and $\iota^{\geq 0} : D^b(X)^{\geq 0} \to D^b(X)$ are equipped with a pair of functors $\tau^{\leq 0}$ and $\tau^{\geq 0}$ right and left adjoints respectively.

Lemma 8.1.

$$\tau^{\leq 0}(\cdots \to A^{-1} \to A^0 \to A^1 \to \cdots) = \cdots \to A^{-1} \to \ker d^0 \to 0\ to \cdots .$$

$$\tau^{\geq 0}(\cdots \to A^{-2} \to A^{-1} \to A^0 \to \cdots) = \cdots \to 0 \to \mathrm{im} d^{-1} \to A^0\ to \cdots .$$

These are the so called truncation functors. The triangle

$$\tau^{\leq 0}A^\bullet \to A^\bullet \to \tau^{\geq 1} \to (\tau^{\leq 0}A^\bullet)[1]$$

is distinguished. The functor $\tau^{\leq 0}\tau^{\geq 0}$ extracts the 0-th cohomology of a complex.

In [BBD] it is shown that any similar setup (called a *t- structure*) the category $D^b(X)^{\leq 0} \cap D^b(X)^{\geq 0}$ is Abelian and moreover, in some particularly nice cases one has

$$D^b(D^b(X)^{\leq 0} \cap D^b(X)^{\geq 0}) \cong D^b(X).$$

Our goal know is to define such a structure that depends on the stratification of a space X and produces an abelian category which has the IC sheaves as its simple objects.

8.1. **Perverse Sheaves.** Given a stratified space X, with strata X_α for $\alpha \in I$ and a perversity p, we first restrict ourselves to $D^b_c(X)$, the full subcategory of $D^b(X)$ whose objects $A^\bullet$ are constructible.

For convenience we will also adjust the notation of perversity to be a function $p : I \to \mathbb{Z}$ that depends only on the dimension of X_α and is thus determined by a function $\tilde{p} : \mathbb{N} \to \mathbb{Z}$ such that: $\tilde{p}(0) = 0$ and if $l \leq k$, then $0 \leq \tilde{p}(l) - \tilde{p}(k) \leq k - l$. I.e. p is zero on the zero dimensional strata and decreases by 0 or 1 each time the dimension of a strata increases by 1.

Under this new definition of perversity, the lower middle is $p(\alpha) = \lceil -\dim_{\mathbb{R}}(X_\alpha)/2\rceil$ and the upper middle is $p(\alpha) = \lfloor -\dim_{\mathbb{R}}(X_\alpha)/2\rfloor$. If we are considering a stratification of a complex variety via even dimensional strata we will refer to either of these as the middle perversity.

Definition 8.2. The perverse t-structure is given by

$${}^pD_c^b(X)^{\leq 0} = \{A^\bullet \in D_c^b(X) | \iota_{X_\alpha}^* A^\bullet \in D_c^b(X_\alpha)^{\leq p(\alpha)} \text{ for all } \alpha \in I\}$$

and

$${}^pD_c^b(X)^{\geq 0} = \{A^\bullet \in D_c^b(X) | \iota_{X_\alpha}^! A^\bullet \in D_c^b(X_\alpha)^{\geq p(\alpha)} \text{ for all } \alpha \in I\}$$

where $\iota_{X_\alpha} \to X$ is the inclusion of X_α. The intersection of these two categories is the category of p-Perverse sheaves.

One may wonder why do we use $\iota^!$ in the definition. This is an artifact of the fact that we want our category well behaved with respect to Verdier duality. In fact, if we let $q = -p(n) - n$ (the dual perversity), then Verdier duality is a functor

$$\mathcal{D} : {}^pD_c^b(X)^{\leq 0} \to {}^qD_c^b(X)^{\geq 0},$$

and

$$\mathcal{D} : {}^pD_c^b(X)^{\geq 0} \to {}^qD_c^b(X)^{\leq 0}$$

In particular, it sends p-perverse sheaves to q-perverse sheaves. If X is a complex variety with an even dimensional stratification, then the categories of perverse sheaves for the two middle perversities are the same and what we generally refer to as perverse sheaves. In particular the category of perverse sheaves is closed under Verdier duality.

Example 8.3.
(1) Consider $\mathbb{C}$ with one stratum $\mathbb{C}$ and the middle perversity. In this case, ${}^pD_c^b(\mathbb{C})^{\leq 0} = D_c^b(\mathbb{C})^{\leq -1}$ and ${}^pD_c^b(X)^{\geq 0} = D_c^b(\mathbb{C})^{\geq -1}$. In this case the category of perverse sheaves is simply the category of constructible complexes concentrated in degree -1. This category is equivalent to the category of constructible sheaves and thus is a category with one simple object $\mathbb{C}_{\mathbb{C}}^\bullet[1]$ which is also self Verdier dual.
(2) Let $\mathcal{O}_{\mathbb{C}}$ be the sheaf of homolorphic functions on $\mathbb{C}$. Consider the complex

$$F = \cdots \to 0 \to \mathcal{O}_{\mathbb{C}} \xrightarrow{D} \mathcal{O}_{\mathbb{C}} \to 0 \to \cdots .$$

Here the first instance of $\prime_{\mathbb{C}}$ is in degree -1 and $D = z\frac{\partial}{\partial z} - \alpha$. $H^{-1}(F) = \ker D$, and so in any small neighbourhood U of a point that is not 0, $Df = 0$ has solutions: $H^{-1}(F)(U) = \mathbb{C}$ with basis a branch of z^α. So $H^{-1}(F)|_{\mathbb{C}^\times}$ is a locally constant sheaf of rank 1. If $\alpha \notin \mathbb{Z}$, then there are no global solutions and moreover no on any neighbourhood containing 0. Thus $H^{-1}(F)|_0 = 0$ in this case. One can also check that we have $H^0(F) = 0$. If $\alpha \in \mathbb{N}$, then there is a global solution z^α. Thus in this case we have $H^{-1}(F) = \mathbb{C}_{\mathbb{C}}$ and $H^0(F) = \mathbb{C}_0$ the skyscraper at 0. Thus this complex is constructable with respect the stratification $\mathbb{C}^\times$, $\{0\}$ and if one checks the costalk at 0, is perverse sheaf with respect to this stratification.

(3) More generally, consider a space X and X_α any open stratum in X. For a pure dimensional variety this will be the top dimensional strata. Since ι_{X_α} is the inclusion of an open set, it follows that $\iota^*_{X_\alpha} = \iota^!_{X_\alpha}$. Then if F is a p-perverse sheaf consider $F|X_\alpha$ Since $\iota^* F \in D^b_c(X)^{\leq p(\alpha)}$ and $\iota^* F \in D^b_c(X)^{\geq p(\alpha)}$, it follows $F|X_\alpha$ is simply a local system on X_α concentrated in degree $p(\alpha)$.

It is possible to rephrase the definition of perverse sheaves to depend less on the stratification that one chooses for X. For this result, we think of the perversity p as a function on $\mathbb{N}$.

Lemma 8.4. *${}^pD^b_c(X)^{\leq 0}$ is the full subcategory of $D^b_c(X)$ where $\dim_{\mathbb{R}}(\mathrm{supp}^j F) \leq k$ for all j and k with $j > p(k)$. ${}^pD^b_c(X)^{\geq 0}$ is the full subcategory of $D^b_c(X)$ where $\dim_{\mathbb{R}}(\mathrm{cosupp}^j F) \leq k$ for all j and k with $j < p(k) + k$. Here $\mathrm{supp}^j F = \{x \in X | H^j(i^*_x F) \neq 0\}$ and $\mathrm{cosupp}^j F = \{x \in X | H^j(i^!_x F) \neq 0\}$ where $i_x : \{x\} \hookrightarrow X$.*

Proof. Since $F \in {}^pD^b_c(X)^{\leq 0}$, it is a constructible complex and thus for all $x \in X_\alpha$ the stalks $H^j(F)_x$ are isomorphic. Moreover $H^j(F)_x = H^j(F_x)$, so $\mathrm{supp}^j F = \mathrm{supp} H^j(F)$ and note that this is contained in a union of strata. Moreover the dimension of these strata is bounded, since $H^j(F)|_{X_\alpha} = 0$ if $j > p(\dim_{\mathbb{R}}(X_\alpha))$. For the other direction, if $\dim_{\mathbb{R}}(\mathrm{supp} H^j(F)) \leq k$ for all j and k with $j > p(k)$, then it follows that for X_α with $j > p(\dim_R X_\alpha)$ we have $H^j(F)_x = 0$ for $x \in X_\alpha$, so $H^j(F)|_{X_\alpha} = 0$. The proof for the other statement is similar. Note that $j < p(k) + k$ is the same as $j > q(k)$ where $p(k) + q(k) = k$. $\square$

In the case of the middle perversity, X with even dimensional strata this becomes even simpler:

Lemma 8.5. *$F \in D^b_c(X)$ is perverse if $\dim_{\mathbb{C}} \mathrm{supp} H^{-j}(F) \leq j$ and $\dim_{\mathbb{C}} \mathrm{cosupp} H^j \leq j$. The second condition is equivalent to $\dim_{\mathbb{C}} \mathrm{supp} H^{-j}(\mathcal{D}F) < j$.*

Remarkably we also have the following result:

Theorem 8.6. *Let X be a complex variety and X_α a even dimensional stratification. If $F \in D^b_c(X)$ is perverse, then $H^j(F) = 0$ for $j < -\dim_{\mathbb{C}} X$ and $j > 0$*

Proof. The second statement is the easiest, since $p(\alpha) \leq 0$, it follows that for any stratum X_α and any $j > 0 \geq p(\alpha)$ we have

$$H^j(F)|X_\alpha = H^j(\iota^*_{X_\alpha} F) = 0.$$

Since the restriction to any strata is 0 the whole sheaf is 0.

For the other direction, consider X with $n = \dim_{\mathbb{C}} X$ and its filtration by strata, i.e. $\cdots X^{\leq 2n-2} \subset X^{\leq 2n} = X$ where $X^{\leq j}$ is the union of all strata of dimension at most j. Let X^j be the union of all j dimensional strata. Let us start by considering $\tau^{\leq -n-1} F$. Since $H^j(\tau^{\leq -n-1} F) \cong H^j(F)$ for $j \leq -n-1$, we only need to show that $H^j(\tau^{\leq -n-1} F) = 0$.

As noted in the example above, $F|_{X^{2n}}$ is a complex that is exact everywhere but at degree $-n$. Thus

$$H^j(F)|_{X^{2n}} = 0$$

for $j < -n$ and

$$H^j(\tau^{\leq -n-1}F)|_{X^{2n}} = 0$$

for all j. This says that the cohomology of this sheaf is only supported on $X^{\leq 2n-2}$. Thus the complex $(\tau^{\leq -n-1}F)|_{X^{\leq 2n-2}}$ is quasi-isomorphic to $\tau^{\leq -n-1}F$. Let $i < 2n$ be the largest such that $X^i \neq \emptyset$, then $X^{\leq 2n-2} = X^{\leq i}$ and X^i is open in this set. Consider $\iota : X^i \to X^{\leq i}$, then we have

$$\begin{aligned}
H^j(F)|_{X^i} &= H^j(\tau^{\leq -n-1}F)|_{X_i} \\
&= H^j((\tau^{\leq -n-1}F)|_{X^{\leq i}})|_{X_i} \\
&= H^j(\iota^*((\tau^{\leq -n-1}F)|_{X^{\leq i}})) \\
&= H^j(\iota^!((\tau^{\leq -n-1}F)|_{X^{\leq i}})) \\
&= H^j(\iota^!\tau^{\leq -n-1}F).
\end{aligned}$$

In particular the second last isomorphism is due to the fact that X^i is open in $X^{\leq i}$ and the last isomorphism is due to the quasi-isomorphism before we apply $\iota^!$.

The final fact we need is that $H^j(\iota^!\tau^{\leq -n-1}F) = H^j(\iota^!F)$ for $j < -n$, since the latter term is always 0 since F is perverse. We have a distinguished triangle

$$\tau^{\leq -n-1}F \to F \to \tau^{\geq -n}F \to (\tau^{\leq -n-1}F)[1].$$

Since $\iota^!$ is a triangulated functor,

$$\iota^!\tau^{\leq -n-1}F \to \iota^!F \to \iota^!\tau^{\geq -n}F \to \iota^!(\tau^{\leq -n-1}F)[1]$$

is a distinguished triangle. Consider the corresponding long exact sequence:

$$\cdots \to H^j(\iota^!\tau^{\leq -n-1}F) \to H^j(\iota^!F) \to H^j(\iota^!\tau^{\geq -n}F) \to \cdots .$$

Recall that

$$\tau^{\geq -n}F = \cdots \to 0 \to \mathrm{im}d^{-n-1} \to F^{-n} \to \cdots ,$$

which is exact in degree $-n-1$. Since $\iota^!$ is a right derived functor of a left exact functor, $\iota^!\tau^{\geq -n}F$ is 0 in degrees $\leq -n-1$ and is exact in degree $-n-1$. Hence $H^j(\iota^!\tau^{\geq -n}F) = 0$ for $j < -n$ and thus $H^j(\iota^!\tau^{\leq -n-1}F) \cong H^j(\iota^!F)$ for $j \leq -n$. $\square$

9. Decomposition

Leray introduced the idea of studying sheaf cohomology of a space X with coefficients in a sheaf $\mathcal{F}$ by taking a map $f : X \to Y$ continuous and studying instead the sheaf cohomology of Y with coefficients in $H^\bullet(Rf_*\mathcal{F})$. Deligne showed that in specific cases one might have the following result:

$$H^i(X, \mathcal{F}) = \oplus_{p+q=i} H^p(Y, H^q(Rf_*\mathcal{F})).$$

Definition 9.1. Let X and Y be smooth varieties. Then a family of projective manifolds is a proper, homolorphic submersion $f : X \to Y$ that factors through $X \hookrightarrow Y \times \mathbb{P}^N$ with fibres smooth projective varieties.

Theorem 9.2 ([D68])**.** *If $f : X \to Y$ is a family of projective manifolds, then*

$$Rf_*\mathbb{C}_X \cong \oplus(H^i Rf_*\mathbb{C}_X)[-i].$$

Corollary 9.3.

$$H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^p(Y, H^q(Rf_*\mathbb{C})).$$

Proof. We have

$$\begin{aligned} H^i(X, \mathbb{C}) &= \mathbb{H}^i(X, \mathbb{C}_X) \\ &= \mathbb{H}^i(Y, Rf_*\mathbb{C}_X) \\ &= \mathbb{H}^i(Y, \oplus(H^j Rf_*\mathbb{C}_X)[-j]) \\ &= \oplus_{p+q=i} \mathbb{H}^p(Y, H^q Rf_*\mathbb{C}_X). \end{aligned}$$

□

The proof of the above theorem will depend on the hard Lefschetz theorem. We begin by recalling the Lefschetz hyperplane theorem and then the hard Lefschetz theorem.

Theorem 9.4 (Lefschetz hyperplane theorem)**.** *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of (complex) dimension n and $Y = H \cap X$ the intersection of a generic hyperplane $H \subset \mathbb{P}^N$. Then the inclusion $\iota : Y \hookrightarrow X$ induces an isomorphism $\iota^* : H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ for $k < n-1$ and an inclusion for $k = n-1$.*

If $\mathcal{H}$ is the hyperplane bundle of X, and $c_1(\mathcal{H})$ its first Chern class, then the map $\eta : H^i(X, \mathbb{C}) \to H^{i+2}(X, \mathbb{C}))$, given by cupping with $c_1(\mathcal{H})$ is the Lefschetz map.

Theorem 9.5 (Hard Lefschetz theorem)**.** *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of (complex) dimension n, then*

$$\eta^i : H^{n-i}(X, \mathbb{C}) \to H^{n+i}(X, \mathbb{C})$$

is an isomorphism.

Now we can prove the previous result:

Proof. Recall that for a sheaf $\mathcal{F}$ on X, $\Gamma(\mathcal{F}) = Hom_{Sh(X)}(\mathbb{C}_X, \mathcal{F})$. In particular,

$$\begin{aligned} H^i(X, \mathbb{C}) &= \mathbb{H}^i(X, \mathbb{C}_X) \\ &= H^i(RHom(\mathbb{C}_X, \mathbb{C}_X)) \\ &= H^0(RHom(\mathbb{C}_X, \mathbb{C}_X[i])) \\ &= Hom_{D^b(X)}(\mathbb{C}_x, \mathbb{C}_X[i]) \end{aligned}$$

In particular, the cup product becomes composition.

Suppose that $X \subset Y \times \mathbb{P}^N$ and $\eta \in H^2(X, \mathbb{C})$ is the first Chern class of the hyperplane bundle. Then $\eta : \mathbb{C}_X \to \mathbb{C}_X[2]$. In particular this induces a map $\eta : H^i(Rf_*\mathbb{C}_X) \to H^{i+2}(Rf_*\mathbb{C}_X)$. We consider the map $\eta^i : H^{n-i}(Rf_*\mathbb{C}_X) \to H^{n+i}(Rf_*\mathbb{C}_X)$, where n is the (complex) dimension of the fibre of f. By hard Lefschetz this is an isomorphism. To see why, let $\iota : \{y\} \hookrightarrow Y$ be the inclusion of a point. Then we consider the induced map on stalks $\eta^i_y : H^{n-i}(\iota^* Rf_*\mathbb{C}_X) \to H^{n+i}(\iota^* Rf_*\mathbb{C}_X)$. Consider the following fibre square

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "\hat{\iota}"] \arrow[d, "\hat{f}"] & X \arrow[d, "f"] \\ \{y\} \arrow[r, "\iota"] & Y \end{tikzcd}$$

Now $R\hat{f} = R\Gamma$ and we have $\iota^* Rf_* = R\Gamma\hat{\iota}^*$ since f is a proper map. But $(\hat{\iota}_y)*$ is just restriction. Thus

$$H^i(\iota^* Rf_*\mathbb{C}_X) = H^i(R\Gamma\hat{\iota}^*\mathbb{C}_X)) = H^i(R\Gamma(\mathbb{C}_{f^{-1}(y)})) = H^i(f^{-1}, \mathbb{C}).$$

Then one can check that η^i_y is just the hard Lefschetz isomorphism.

We apply the lemma below to the case of a category $\mathcal{A} = Sh(Y)$, $X = Rf_*\mathbb{C}_X \in \mathcal{A}[n]$ and η the Lefschetz operator and the result follows. □

Lemma 9.6. *Let $\mathcal{A}$ be an abelian category, $X^\bullet \in D^b(\mathcal{A})$ and $\eta : X^\bullet \to X^\bullet[2]$ such that $\eta^i : H^{-i}(X^\bullet) \to H^i(X^\bullet)$ is an isomorphism for each i, then $X^\bullet \cong_{D(\mathcal{A})} \oplus H^i(X^\bullet)[-i]$.*

[vdB]. Let d be an integer such that $H^j(X^\bullet) = 0$ for $|j| > d$. We induct down on d. Consider

$$\alpha : H^{-d}(X^\bullet) \cong \tau^{\leq 0}(X^\bullet[-d]) \to X^\bullet[-d] \to X^\bullet[d] \to \tau^{\geq 0} X^\bullet[d] \cong H^d(X^\bullet).$$

One can then check that we get maps $X \to H^{-d}(X)[d] \oplus H^d(X)[-d]$ and $H^{-d}(X)[d] \oplus H^d(X)[-d] \to X$ whose composition is the identity on $H^{-d}(X)[d] \oplus H^d(X)[-d]$. This forces X to factor $X \cong H^{-d}(X)[d] \oplus H^d(X)[-d] \oplus X'$ where $H^j(X') = 0$ for $|j| \geq d$. □

It should be noted that when X and Y are also quasi-projective Deligne shows that

Theorem 9.7. [D71] *$H^i(Rf_*\mathbb{C}_X)$ are semisimple local systems on Y.*

9.1. **Perverse Cohomology.** We would like to be able to extend the above idea to any proper map $f : X \to Y$ of complex algebraic varieties where X is non singular. Since we do not have

$$Rf_*\mathbb{C}_X = \oplus H^i(Rf_*\mathbb{C}_X)[-i],$$

in this situation we need to replace the right hand side with something else.

In particular, associated to ${}^pD^b_c(X)^{\leq 0}$ and ${}^pD^b_c(X)^{\geq 0}$ are perverse truncation functors. For their full definition see [KS] 10.2. It is sufficient to think of them as

$${}^p\tau^{\leq 0} : D^b_c(X) \to {}^pD^b_c(X)^{\leq 0}$$

and

$${}^p\tau^{\geq 0} : D^b_c(X) \to {}^pD^b_c(X)^{\geq 0}.$$

In particular, for a strata X_α with $\dim_{\mathbb{C}} X_\alpha = n$ and $\iota : X_\alpha \hookrightarrow X$, the functors should satisfy:

$$\iota^{*p}\tau^{\leq 0}F^\bullet = \tau^{\leq -n}\iota^*F^\bullet$$

and

$$\iota^{!p}\tau^{\geq 0}F^\bullet = \tau^{\geq -n}\iota^!F^\bullet.$$

As usual, we define ${}^p\tau^{\leq i} = [-i] \circ {}^p\tau^{\leq 0} \circ [i]$ and ${}^p\tau^{\geq i} = [-i] \circ {}^p\tau^{\geq 0} \circ [i]$.

Also we should have a distinguished triangle in $D^b_c(X)$ of the form

$${}^p\tau^{\leq 0}F^\bullet \to F^\bullet \to \iota^{!p}\tau^{\geq 1}F^\bullet \to ({}^p\tau^{\leq 0}F^\bullet)[1]$$

Definition 9.8. The 0-th *perverse cohomology*, denoted ${}^pH^0 : D^b_c(X) \to Perv(X)$ is the composition ${}^p\tau^{\leq 0} \circ {}^p\tau^{\geq 0}$.

It is a cohomological functor, that is given a distinguished triangle in $D^b_c(X)$, applying ${}^pH^0$ gives a long exact sequence in $Perv(X)$. It is typical to now define ${}^pH^i = {}^pH^0 \circ [i]$.

One can check that

Lemma 9.9. *$\mathcal{D} \circ {}^p\tau^{\leq k} \circ \mathcal{D} \cong {}^p\tau^{\geq -k}$ and $\mathcal{D} \circ {}^pH^k \circ \mathcal{D} \cong {}^pH^{-k}$*

Definition 9.10. We say that $K^\bullet \in D^b_c(X)$ is p-split if $K^\bullet \cong \oplus {}^pH^i(K^\bullet)[-i]$

Our goal now is prove the following theorem (and more!);

Theorem 9.11 ([BBD])**.** *If $f : X \to Y$ is a proper map of algebraic varieties where X is smooth, then $Rf_*\mathbb{C}_X$ is p-split, ${}^pH^i(Rf_*\mathbb{C}_x)$ is semi-simple in $Perv(Y)$ and there exists $\eta^i : {}^pH^{-i}(Rf_*\mathbb{C}_X[\dim_{\mathbb{C}} X]) \to {}^pH^i(Rf_*\mathbb{C}_X[\dim_{\mathbb{C}} X])$ an isomorphism in $Perv(Y)$.*

Note 9.12. Given a map $f : A^\bullet \to B^\bullet$, two perverse sheaves, let $C^\bullet = cone(f)$. Then consider the long exact sequence associated to that triangle. Since ${}^pH^0(A^\bullet[i]) = 0$ when $i \neq 0$, the associated long exact sequence of perverse sheaves gives a short exact sequence:

$$0 \to {}^pH^{-1}(C^\bullet) \to A^\bullet \to B^\bullet \to {}^pH^0(C^\bullet) \to 0.$$

I.e. ${}^pH^{-1}(cone(f))$ is the kernel of f and ${}^pH^0(cone(f))$ is the cokernel of f.

In particular we also see that any distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D^b_c(X)$ of perverse sheaves gives a short exact sequence of perverse sheaves.

9.2. **Hopf Fibration.** Let us explore some examples of $f : X \to Y$ where

$$Rf_*\mathbb{C}_X \cong \oplus(H^iRf_*\mathbb{C}_X)[-i]$$

fails to be true. We will actually check that

$$H^i(X,\mathbb{C}) \ncong \oplus_{p+q=i}H^p(Y, H^q(Rf_*\mathbb{C}_X)).$$

Let $M = \mathbb{C}^2 - \{(0,0)\}$, it is acted on by $\mathbb{C}^\times$ via dialations and by $\mathbb{Z}$ where the generator of $\mathbb{Z}$ acts by multiplication by $\frac{1}{2}$.

(1) $M \to M/\mathbb{C}^\times = \mathbb{P}^1$, this is the $\mathbb{C}^\times$ bundle of the complex line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$.
(2) The Hopf fibration $S^3 \to S^2$ by restriction to the 3-sphere on the left.
(3) The Hopf surface $M/\mathbb{Z}$ and there is a natural map to $\mathbb{P}^1$ which is a proper holomorphic submersion with elliptic curves as fibres.

If $f : X \to S^2$ where f is taken as any of the three maps above, we have

$$H^i(X,\mathbb{C}) \ncong \oplus_{p+q=i}H^p(S^2, H^q(Rf_*\mathbb{C}_X)).$$

Why? We have

$$b_1(X) = \sum_{p+q=1} \dim H^p(S^2, H^q(Rf_*\mathbb{C}_X)).$$

Moreover $H^q(Rf_*\mathbb{C}_X)$ are locally constant since S^2 is simply connected. So

$$b_1(X) = \sum_{p+q=1} \dim H^p(S^2,\mathbb{C}) \dim H^q(Rf_*\mathbb{C}_X) \geq b_0(S^2) \dim H^1(Rf_*\mathbb{C}_X) = \dim H^1(\text{fibre}).$$

But in case 1 and 2 we have $b_1(X) = 0$ while $b_1(\text{fibre}) = 1$ and in case 3 we have $b_1(X) = 1$ while $b_1(\text{fibre}) = 2$

In case 1, the map is a complex algebraic map which is Zariski locally trivial but not proper.

In case 2, the map is a real algebraic proper submersion.

In case 3, the map is a holomorphic proper submersion.

In the case of the Hopf fibration $f : S^3 \to S^2$, what is $Rf_*\mathbb{C}_{S^3}$? From the truncation distinguished triangle, we have a short exact sequence of complexes:

$$0 \to H^0(Rf_*\mathbb{C}_{S^3}) \to Rf_*\mathbb{C}_{S^3} \to H^1(Rf_*\mathbb{C}_{S^3})[-1] \to 0.$$

One can check that $H^0(Rf_*\mathbb{C}_{S^3}) = H^1(Rf_*\mathbb{C}_{S^3}) = \mathbb{C}_{S^2}$ via taking stalks since S^2 is simply connected. Thus since $Rf_*\mathbb{C}_{S^3}$ doesn't split, it is the unique (up to isomorphism in $D^b_c(S^2)$) complex making $0 \to \mathbb{C}_{S^2} \to \bullet \to \mathbb{C}_{S^2}[-1] \to 0$ exact and non-split.

9.3. **Simple objects in Perv(X).** Recall that in an abelian category, a *subobject* of an object A in the category is an object B along with a monomorphism $B \to A$. Moreover an object is *simple* if it has no subobjects other than 0 and itself (and is not 0 itself). A *semisimple* object is the direct sum of simple objects.

Note 9.13. As a warning, set theoretic intuitions of subobjects/quotients can fail for $Perv(X)$. Let X be the unit disc, stratified by the origin o and its complement $X^\times$. Consider the two functions $i : o \hookrightarrow X \hookleftarrow X^\times : j$. Then $j_!\mathbb{C}_{X^\times}$ is the sheaf on X that has sections $\mathbb{C}$ over any connected open set not containing o and sections 0 over those connected open sets containing o. $i_*\mathbb{C}_o$ is skyscraper sheaf at o. In the category of complexes (or indeed sheaves) we have a short exact sequence

$$0 \to j_!\mathbb{C}_{X^\times} \to \mathbb{C}_X \to i_*\mathbb{C}_o \to 0.$$

This gives rise to a distinguished triangle in the derived category.

One particular rotation has all 3 complexes perverse:

$$\cdots \to i_*\mathbb{C}_o \to (j_!\mathbb{C}_{X^\times})[1] \to \mathbb{C}_X[1] \to \cdots .$$

So this gives rise to a short exact sequence of perverse sheaves after applying ${}^pH^0$. This implies that $\mathbb{C}_X[1]$ is $(j_!\mathbb{C}_{X^\times})[1]/i_*\mathbb{C}_o$ but set theoretically $(j_!\mathbb{C}_{X^\times})[1]$ is a subsheaf of $\mathbb{C}_X[1]$.

Lemma 9.14. *The category $Perv(X)$ is Noetherian. That is, any ascending chain of subobjects of an object $P^\bullet \in Perv(X)$ stabilizes.*

To see this one first argues that the category of constructable sheaves is Noetherian. Then use the fact $Perv(X)$ is abelian to show that eventually in any ascending chain, the homology of the maps in the chain become isomorphisms.

By applying Verdier duality, we then immediately get

Lemma 9.15. *The category $Perv(X)$ is Artinian.*

Thus we have the following theorem

Theorem 9.16. *Any object in $Perv(X)$ admits a finite decreasing filtration (composition series),*

$$P^\bullet = Q_1^\bullet \subset Q_2^\bullet \subset \cdots \subset Q_k^\bullet = 0,$$

where $Q_i^\bullet/Q_{i-1}^\bullet$ are simple perverse sheaves.

We will prove this once we have an idea of what the simple objects are.

9.4. **Intermediate (middle) extension.** For a stratified space X, let U be an open stratum and Z be its complement. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$.

There are 8 maps of interest to us on the level of derived categories. The following is a chart of their properties:

$$\begin{array}{cc} j_!,\ i^* & \text{right } t\text{-exact} \\ j^! = j^*,\ i_* = i_! & t\text{-exact} \\ Rj_*,\ i^! & \text{left } t\text{-exact} \end{array}$$

Recall that $F : D^b_c(X) \to D^b_c(Y)$ is left exact if $F({}^pD^b_c(X)^{\geq 0}) \subset {}^pD^b_c(Y)^{\geq 0}$ and is right t-exact if $F({}^pD^b_c(X)^{\leq 0}) \subset {}^pD^b_c(Y)^{\leq 0}$.

Let ${}^pF = {}^pH^0 \circ F$, then as functors on the category of perverse sheaves we have

$$\begin{array}{cc} {}^pj_!,\ {}^pi^* & \text{right exact} \\ j^! = j^*,\ i_* = i_! & \text{exact} \\ {}^pj_*,\ {}^pi^! & \text{left exact} \end{array}$$

For any sheaf $Q^\bullet \in D^b(X)$, there is a natural map $\alpha : j_!Q^\bullet \to Rj_*Q^\bullet$. In particular consider if $Q^\bullet \in Perv(X)$, then we apply ${}^pH^0$ to this map. Since ${}^pH^0(\alpha) : {}^pj_!Q^\bullet \to {}^pj_*Q^\bullet$ is a map of perverse sheaves it has an image, which we will denote $j_{!*}Q$. This functor is called the (Goresky-MacPherson) intermediate (middle) extension functor.

In general, a complex $F^\bullet \in D^b_c(X)$ is an extension of a complex $Q^\bullet \in D^b_c(U)$ if $F^\bullet|_U = Q^\bullet$. In particular, adjunction always gives morphisms $j_!Q^\bullet \to F \to Rj_*Q^\bullet$. Thus in some sense $j_!Q^\bullet$ is the smallest extension of $Q^\bullet$ and $Rj_*Q^\bullet$ is the largest. If we consider ${}^pj_!Q^\bullet$, one can show that this is the unique extension $F^\bullet$ of $Q^\bullet$ such that $i^*F^\bullet \in {}^pD^b_c(Z)^{\leq -2}$ and $i^!F^\bullet \in {}^pD^b_c(Z)^{\geq 0}$. Similarly, one has that ${}^pj_*Q^\bullet$ is the unique extension $F^\bullet$ of $Q^\bullet$ such that $i^*F^\bullet \in {}^pD^b_c(Z)^{\leq 0}$ and $i^!F^\bullet \in {}^pD^b_c(Z)^{\geq 2}$.

Note 9.17. Since $F^\bullet$ is a perverse sheaf, it satisfies the usual (co)support conditions. I.e. that the i-th cohomology of $F^\bullet$ is supported only on strata of complex dimension less than or equal to $-i$ and is cosupported only on strata of complex dimension less than or equal to i.

The condition $i^*F^\bullet \in {}^pD^b_c(Z)^{\leq -2}$ says that on the strata of Z, the sheaf F satisfies some more restrictive support conditions, namely that i-th cohomology of $F^\bullet$ is supported only on strata of Z of complex dimension less than or equal to $-i-2$. The condition $i^!F^\bullet \in {}^pD^b_c(Z)^{\geq 0}$ is satisfied by any perverse sheaf.

Lemma 9.18. *$j_{!*}Q^\bullet$ is the unique extension of $Q^\bullet$ to X such that $i^*j_{!*}Q^\bullet \in {}^pD^b_c(Z)^{\leq -1}$ and $i^!j_{!*}Q^\bullet \in {}^pD^b_c(Z)^{\geq 1}$.*

See [BBD] 1.4.24 for the proof of this fact and 1.4.22 for the previous one. The following is result 1.4.25:

Theorem 9.19. *$j_{!*}Q^\bullet$ is the unique perverse extension of $Q^\bullet$ to X that has no non-trivial subobjects or quotients supported on Z.*

Proof. Suppose that $F^\bullet$ is an extension of $Q^\bullet$ on X. Then $i^*F^\bullet \in {}^pD^b_c(Z)^{\leq 0}$ and ${}^pi^*F^\bullet = 0$ if and only if $i^*F^\bullet \in {}^pD^b_c(Z)^{\leq -1}$. Similarly for $i^!F^\bullet$.

Now, $i_*{}^p i^! F^\bullet$ is the largest subject of $F^\bullet$ supported on Z and $i_*{}^p i^* F^\bullet$ is the largest quotient supported on Z.

In general we have two distinguished triangles,

$$j_! j^* F^\bullet \to F^\bullet \to i_* i^* F^\bullet \to (j_! j^* F^\bullet)[1]$$

and

$$i_* i^! F^\bullet \to F^\bullet \to j_* j^* F^\bullet \to (i_* i^! F^\bullet)[1].$$

Applying ${}^pH^0$ we get long exact sequences,

$$\cdots \to {}^p j_! j^* F^\bullet \to F^\bullet \to i_*{}^p i^* F^\bullet \to 0 \to \cdots$$

and

$$\cdots \to 0 \to i_*{}^p i^! F^\bullet \to F^\bullet \to {}^p j_* j^* F^\bullet \to \cdots .$$

□

Let U be any stratum, then we consider $\hat{j} : U \to \bar{U}$ and $i : \bar{U} \to X$ where $i_* : Perv(\bar{U}) \to Perv(X)$ is extension by 0, then $j_{!*} : Perv(U) \to Perv(X)$ given by $i_* \circ \hat{j}$. It satisfies the same property, that there are no non-trivial subobjects or quotients supported on $X - U$.

Note that for a single stratum U, the category $Perv(U)$ is simply the equivalent to category of local systems concentrated in degree $-\dim_{\mathbb{C}} U$.

Also, it should be noted that although $j_{!*}$ preserves both monomorphisms and epimorphisms, it is not in general an exact functor.

Theorem 9.20. *The category $Perv(X)$ is Artinian and the simple objects are given by a pair of stratum U and $\mathcal{L}$ local system on U via $j : U \to X$:*

$$j_{!*}(L[\dim_{\mathbb{C}} U]).$$

Proof. First lets check that $j_{!*}(L[\dim_{\mathbb{C}} U])$ is indeed simple. Suppose that $P^\bullet \to j_{!*}(L[\dim_{\mathbb{C}} U])$ is monomorphism. In particular, the restriction functor from X to a open union of strata is exact and sends perverse sheaves to perverse sheaves. Thus the restriction of the above map to $X - \bar{U}$ is still injective, but the co-domain is now 0, so $P^\bullet$ is supported in $\bar{U}$. On the other hand the support of $P^\bullet$ must be all of $\bar{U}$ since on $\bar{U}$, the sheaf $j_{!*}(L[\dim_{\mathbb{C}} U])$ admits no non-trivial subobjects or quotients.

In particular, the restriction of $P^\bullet$ to U must be a shifted local system and in particular it is $L[\dim_{\mathbb{C}} U]$ since there is a monomorphism from $P^\bullet|_U \to L[\dim_{\mathbb{C}} U]$ and L is simple. Thus $P^\bullet / j_{!*}(L[\dim_{\mathbb{C}} U])$ is supported away from U which is impossible, since $j_{!*}(L[\dim_{\mathbb{C}} U])$ admits no such quotients.

Now we prove that every object has a composition series such that the quotients are all simple objects of the form $j_{!*}(L[\dim_{\mathbb{C}} U])$.

We use Notherian induction. The fact is true for any closed stratum X_α of X since the only perverse sheaves supported on X_α are isomorphic to $L[\dim X_\alpha]$ for L a local system on X_α. In particular, one takes a composition series for L and uses the fact that $j_{!*}$ preserves monics to construct a composition series for $L[\dim X_\alpha]$.

Let us now assume that it is true for any closed union of strata of that is not all of X. Let $P^\bullet \in Perv(X)$, there is a top dimensional open stratum U such that $P^\bullet|_U = L[\dim_{\mathbb{C}} U]$ where L is a local system. If not, then $P^\bullet$ has such a composition series by induction.

Let $j : U \to X$, then $j^! = j^*$ so from the adjuctions for ! and $*$ we have natural maps

$$j_! j^* P^\bullet \to P^\bullet \to Rj_* j^* P^\bullet.$$

Note that $j^* P^\bullet = P^\bullet|_U$, so when we apply ${}^pH^0$, we have the following commutative diagram:

$$\begin{tikzcd}
{}^pH^0 j_! j^* P^\bullet \arrow[rr] \arrow[d] \arrow[drr] & & P^\bullet \arrow[d, "b"] \\
j_{!*} L[\dim_{\mathbb{C}} U]\{y\} \arrow[rr] & & {}^pH^0 j_* j^* P^\bullet
\end{tikzcd}$$

In particular this shows that $j_{!*}L[\dim_C U]$ is a subobject of the image of b. In particular, if $c : P^\bullet \to \Im b / j_{!*} L[\dim_{\mathbb{C}} U]$, then $j_{!*} L[\dim_{\mathbb{C}} U] = \ker c / \ker b$. In particular, $\ker b \subset \ker c \subset P^\bullet$ with quotients $\ker b$, $j_{!*}L[\dim_{\mathbb{C}} U]$ and $P^\bullet / \ker c$. The middle has a composition series as desired and the left and right are supported on $X - U$ and so by induction also have the desired composition series. $\square$

Lemma 9.21. *If $Q^\bullet \in Perv(U)$ is self dual, so is $j_{!*}Q^\bullet$.*

Proof. Let $\alpha : j_! Q^\bullet \to Rj_* Q^\bullet$, then we have $\mathcal{D}(\mathrm{im}(\alpha)) = \mathrm{im}(\mathcal{D}(\alpha))$, but $\mathcal{D}(\alpha) : \mathcal{D}(Rj_* Q^\bullet) \to \mathcal{D}(j_! Q^\bullet)$. We commute the $\mathcal{D}$ past the Rj_* and $j_!$ to get $\mathcal{D}(\alpha) : j_! \mathcal{D}(Q^\bullet) \to Rj_* \mathcal{D}(Q^\bullet)$. In particular we will have $\mathcal{D}(\alpha) = \alpha$. $\square$

Since any $L[\dim_C U]$ is self dual, we immediately conclude that all simple perverse sheaves are self dual.

There is a direct construction of the functor $j_{!*}$. Let U_l be the union of the strata in $\bar{U}$ whose dimension is greater than or equal to l. Note that U_l is open and we have a sequence of maps $j_i : U_i \to U_{i-1}$. Let $m = \dim_{\mathbb{C}} U$, then

Theorem 9.22.

$$j_{!*} Q^\bullet = \tau^{\leq -1} R j_{1*} \cdots \tau^{\leq -m} R j_{m*} Q^\bullet.$$

So our goal now is to prove:

Theorem 9.23 ([BBD])**.** *If $f : X \to Y$ is a proper map of algebraic varieties, and $S^\bullet$ a simple perverse sheaf on X, then $Rf_* S^\bullet$ is p-split, ${}^pH^i(Rf_* S^\bullet)$ is semi-simple in $Perv(Y)$ and there exists $\eta^i : {}^pH^{-i}(Rf_* S^\bullet) \to {}^pH^i(Rf_* S^\bullet)$ an isomorphism in $Perv(Y)$.*

In particular we will examine the case when $f : X \to Y$ is semi-small.

Definition 9.24. Let $f : X \to Y$ a proper map of algebraic varieties. f is semi-small if for each strata Y_α and $y \in Y_\alpha$ we have

$$\dim_{\mathbb{C}} f^{-1}(y) \leq \frac{1}{2}(\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} Y_\alpha).$$

9.5. **IC Sheaves.**

Definition 9.25. If L is a local system on a stratum U of X, we define $IC_X(L) = j_{!*}L[\dim_{\mathbb{C}} U]$.

Example 9.26. If $X = \mathbb{C}$ stratified by 0 and $\mathbb{C} - \{0\}$ and L is a local system on $\mathbb{C} - \{0\}$. Then $IC_X(L) = j_{!*}(L[1]) = \tau^{\leq -1}(Rj_*L[1]) = (j_*L)[1]$.

Note 9.27. It is the case that $IC_X(L)$ is the unique self dual perverse sheaf supported on $\bar{U}$ such that $IC_X(L)|_U = L[\dim_{\mathbb{C}} U]$ and $\dim \operatorname{supp}^{-i}(IC_X(L)) < i$ and $\dim \operatorname{cosupp}^i(IC_X(L)) < i$ for all $i < \dim_{\mathbb{C}} U$.

Theorem 9.28. [GM83] *Let X be a pure dimensional stratified variety with one top dimensional strata U. Let IC_X^{GM} be the complex where $(IC_X^{GM})^k(V)$ is the collection of allowable Borel-Moore $-k$-chains on V. Then, $IC_X^{GM}[-n] \cong IC_X(\mathbb{C}_U)$ where U. We will denote $IC_X(\mathbb{C}_U)$ by IC_X.*

Note 9.29. The fact that the Goresky-MacPherson IC sheaf is a shift of what we are calling an IC sheaf. Our notation emphasizes the self duality of the IC sheaf and is the more modern notation. Be careful when reading older papers since this can cause some confusion.

Definition 9.30. On an complex variety stratified pure variety X with $\dim_{\mathbb{C}} = n$, the intersection cohomology with coefficients in a local system L is defined to be $IH(L)^k = \mathbb{H}^{k-n}(X, IC_X(L)) = \mathbb{H}^k(X, IC_X(L)[-n])$. In particular we have $IH^k = \mathbb{H}^k(X, IC_X[-n])$. Similarly we have $IH(L)_k \cong IH(K)^{2n-k}$ and intersection (co)homology with compact supports is analogously defined.

Theorem 9.31. *Let X be an irreducible complex n-dimensional variety over $\mathbb{C}$ with isolated singularities. Let U be the smooth part of X, then $IH^k(X) = H^k(X)$ for $k < n$, $IH^n(X) = \operatorname{im} H^n(X) \to H^n(U)$ and $IH^k(X) = H^k(X)$ for $k > n$.*

Proof. Set $S = X - U$, then S and U form stratification of X and let $j : U \hookrightarrow X$. Applying our formula,

$$IC_X = j_{!*}\mathbb{C}_U[n] = \tau^{\leq -1}(Rj_*\mathbb{C}_U[n]),$$

but then $IC_X[-n] = \tau^{\leq n-1}(Rj_*\mathbb{C}_U)$ and it fits into a distinguished triangle

$$IC_X[-n] \to Rj_*\mathbb{C}_U \to \tau^{\geq n}Rj_*\mathbb{C}_U \to IC_X[-n+1].$$

Applying hypercohomology gives a long exact sequence where $\mathbb{H}^i(X, \tau^{\geq n}Rj_*\mathbb{C}_U) = 0$ for $i < n$, which shows that $IH^k(X) \cong H^k(U)$ for $k < n$.

$H^0(IC_X[-n]) = H^0(Rj_*\mathbb{C}_U) = \mathbb{C}_X$. In particular we have a morphism $\mathbb{C}_X \to IC_X[-n]$, the composition with $IC_C[-n] \to Rj_*\mathbb{C}_U$ is exactly the left part adjunction triangle:

$$\mathbb{C}_X \to Rj_*\mathbb{C}_U \to i_! i^! C_X[1] \to \mathbb{C}_X[1],$$

where $\mathbb{C}_U = j^*\mathbb{C}_X$ and $i : S \hookrightarrow X$. In particular this gives a morphism between the above two triangles and applying hypercohomology and the 5-lemma will given the remaining part of the result. □

Example 9.32. Let $G = SL_3(\mathbb{C})$ and consider the nilpotent cone $\mathcal{N} \in \mathfrak{sl}_3$. $\mathcal{N}$ is a algebraic variety and there exists a stratification by the nilpotent orbits. These orbits are orbits under the conjugation action via G, which in this case splits $\mathcal{N}$ up by Jordan type.

There are 3 orbits, $\mathcal{O}_0$ (3 Jordan blocks), $\mathcal{O}_m$ (2 Jordan blocks) and $\mathcal{O}_{reg}$ (1 Jordan block). These have complex dimension 0, 4 and 6 respectively. There is a resolution of singularities $\tilde{\mathcal{N}}$, the Springer resolution, which is formed of pairs of $n \in \mathcal{N}$ and flags in $\mathbb{C}^3$ fixed by n ($0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3$ and $nV_i \subset V_{i-1}$).

The map $f : \tilde{\mathcal{N}} \to \mathcal{N}$ is semi-small: over any point in $\mathcal{O}_{reg}$, the fibre has dimension 0. Over $\mathcal{O}_0$, the fibre is 3 dimensional and $3 \leq \frac{1}{2}(6-0)$. One can check that the dimension of the fibre over $\mathcal{O}_m$ is 1 dimensional.

If the decomposition theorem is true, then $Rf_*\mathbb{C}_{\mathcal{N}}[6]$ is a direct sum of IC sheaves. First, lets restrict to $\bar{\mathcal{O}}_m$. Its resolution has a slightly simpler form: We pair a nilpotent n with a lines in $\mathbb{C}^3$ that contains the image of n. Call this $\tilde{\mathcal{O}}_m$, this resolution is also semi-small.

Consider $Rf_*\mathbb{C}_{\tilde{\mathcal{O}}_m}[4]$ is also a direct sum of IC sheaves. In particular we have

$$(Rf_*\mathbb{C}\tilde{\mathcal{O}}_m[4])|_{\mathcal{O}_m} = \mathbb{C}_{\mathcal{O}_m}[4]$$

and

$$(Rf_*\mathbb{C}\ \tilde{\mathcal{O}}_m[4])|_{\mathcal{O}_0} = H^\bullet(\mathbb{PC}^2)[4].$$

In particular, over $\mathcal{O}_0$ the sheaf has stalks $\mathbb{C}$ in degrees 0, -2 and -4 and over $\mathcal{O}_m$ in degree -4.

On $\bar{\mathcal{O}}_m$ there is only one IC sheaf that can have support (and hence a stalk) in degree 0. That is $IC_{\bar{\mathcal{O}}_m}(\mathbb{C}_{\mathcal{O}_0})$. If we examine the remaining stalks, only one appears in a degree which is the negative of a stratum, i.e. in degree -4. The orbit $\mathcal{O}_m$ is simply connected, so there is only one simple IC sheaf on the orbit. Hence

$$Rf_*\mathbb{C}_{\tilde{\mathcal{O}}_m}[4] = IC_{\bar{\mathcal{O}}_m}(\mathbb{C}_{\mathcal{O}_m}) \oplus IC_{\bar{\mathcal{O}}_m}(\mathbb{C}_{\mathcal{O}_0}).$$

Applying the same reasoning to $Rf_*\mathbb{C}_{\mathcal{N}}[6]$, we see that it is a direct sum

$$Rf_*\mathbb{C}_{\mathcal{N}}[6] = IC_{\mathcal{N}}(\mathbb{C}_{\mathcal{O}_{reg}}) \oplus IC_{\mathcal{N}}(\mathbb{C}_{\mathcal{O}_m})^{\oplus 2} \oplus IC_{\mathcal{N}}(\mathbb{C}_{\mathcal{O}_0}).$$

9.6. **Semi-small maps.**

Definition 9.33. A proper map $f : X \to Y$ is called weakly stratified if for each stratum Y_α of Y, $f|_{f^{-1}(Y_\alpha)}$ is a locally trivial fibre bundle. f is a stratified map if $f^{-1}(Y_\alpha)$ is a union of strata.

It should be noted here that in general to preserve constructability of complexes of sheaves under the various functors we have defined requires that the underlying map f be stratified. If one starts with a weakly stratified map $f : X \to Y$, one can attempt to refine the stratification on X to stratify the map and refining stratifications only enlarges the category of perverse sheaves. In particular $f : X \to Y$ a resolution of singularities is not a stratified map but $Rf_*\mathbb{C}_X$ is constructible since f is weakly stratified and the stratification on X can be refined.

We will assume that maps are now stratified. Let us restate the definition of begin semi-small:

Definition 9.34. Let $f : X \to Y$ a proper map of algebraic varieties. f is semi-small if for each strata Y_α and $y \in Y_\alpha$ we have

$$\dim_{\mathbb{C}} f^{-1}(y) \leq \frac{1}{2}(\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} Y_\alpha).$$

Lemma 9.35. *Given a proper, semi-small map $f : X \to Y$ between algebraic varieties, where X is a non singular variety with $\dim_{\mathbb{C}} X = n$, $Rf_*\mathbb{C}_X[n]$ is perverse.*

Proof. We need to check that $Rf_*\mathbb{C}_X[n]$ satisfies the conditions of being a perverse sheaf.

First, note that

$$\mathcal{D}Rf_*\mathbb{C}_X[n] = Rf_!\mathcal{D}\mathbb{C}_X[n] = Rf_!\mathbb{C}_X[n] = Rf_*\mathbb{C}_X[n].$$

Here we have $f_* = f_!$ since f is proper. Thus we only need check that $\dim_{\mathbb{C}} \operatorname{supp}^{-i}(Rf_*\mathbb{C}_X[n]) \leq i$ since the cosupport condition will then be satisfied by duality.

Let $i_y : \{y\} \hookrightarrow Y$, then

$$H^j(i_y^* Rf_*\mathbb{C}_X) = \mathbb{H}^j(f^{-1}(y), \mathbb{C}_X[n]|_{f^{-1}y}) = H^{j+n}(f^{-1}(y)).$$

In particular, if $H^j(i_y^* Rf_*\mathbb{C}_X) \neq 0$, then $j + n \leq 2\dim_C f^{-1}(y) \leq n - \dim_{\mathbb{C}} Y_\alpha$ where $y \in Y_\alpha$ since f is semi-small. In particular, since f is semi-small, $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$. We conclude that $H^j(i_y^* Rf_*\mathbb{C}_X) \neq 0$ only on strata Y_α with $\dim_{\mathbb{C}} Y_\alpha \leq -j$. $\square$

Note 9.36. The above is also true if we replace $\mathbb{C}_X$ by any local system on X. In particular, $\mathbb{H}j + n(f^{-1}, L|_{f^{-1}(y)})$ is singular cohomology with local coefficients in the local system $L|_{f^{-1}(y)}$, which vanishes in degrees above twice the dimension. See [B] III.1. If one puts further conditions on f, *stratified semi-small* (see [MV]), then one can also obtain that the Rf_* sends perverse sheaves to perverse sheaves.

Definition 9.37. In the setup above, a stratum Y_α is called *relevant* if $\dim_{\mathbb{C}} f^{-1}(y) = \frac{1}{2}(\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} Y_\alpha)$.

Notice that in the above argument, if Y_α is a relevant stratum of dimension d, and $i : Y_\alpha \to Y$, then $H^{-d}(i^* Rf_* \mathbb{C}_X) \neq 0$. Since f is proper, $f^{-1}(y)$ is compact and the stalk, $H^{n-d}(f^{-1}(y))$, the top dimensional cohomology of $f^{-1}(y)$, has basis given by the top dimensional irreducible components of $f^{-1}(y)$ (i.e. it is isomorphic to $H^{BM}_{n-d}(f^{-1}(y)))$).

In particular the local system $H^{-d}(i^* Rf_* \mathbb{C}_X)$ gives a representation of $\pi_1(Y_\alpha)$ on $H^{BM}_{n-d}(f^{-1}(y)))$ by permuting the irreducible components. Thus as a permutation representation, it is semisimple.

Definition 9.38. Given a relevant stratum Y_α, a simple local system is relevant if it appears in the decomposition of $H^{-d}(i^* Rf_* \mathbb{C}_X)$.

Lemma 9.39. *On a relevant stratum Y_α, the local system $\mathbb{C}_{Y_\alpha}$ is always relevant.*

Proof. The multiplicity of $\mathbb{C}_{Y_\alpha}$ is exactly the multiplicity of the trivial representation in the representation of $\pi_1(Y_\alpha)$ on $H^{BM}_{n-d}(f^{-1}(y)))$. But this is equal to the number of orbits of the permutation action of $\pi_1(Y_\alpha)$ on the irreducible components. □

Theorem 9.40 (Decomposition for semi-small maps). *If $f : X \to Y$ is proper, semi-small and X is non-singular, then*

$$Rf_* \mathbb{C}_X \cong \oplus_L IC_X(L)^{\oplus d_L}.$$

where L are relevant local system and d_L is its multiplicity.

10. Springer Resolution/Correspondence

This is derived from the corresponding sections of [CG] and [G]. Let G be a complex, semisimple Lie group, $\mathfrak{g}$ is Lie algebra, B a Borel subgroup and T a maximal torus in G. Let $\mathfrak{b}$ be the corresponding Borel subalgebra, $\mathfrak{h}$ the corresponding Cartan subalgebra. The Weyl group W is $N_G(T)/T$.

Example 10.1. If G is SL_n we can take B and T to be the upper triangular and diagonal matrices in G respectively. $\mathfrak{g}$ is then the set of traceless matrices and $W \cong S_n$.

Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, that is the set of all nilpotent elements in $\mathfrak{g}$. Note that an element x is nilpotent if $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent (this is the map which takes the Lie bracket with x). In the case of $\mathfrak{sl}_n$ this just means that x is nilpotent.

Springer associated to $\mathcal{N}$, a resolution $f : \tilde{\mathcal{N}} \to \mathcal{N}$ called the Springer resolution. $\tilde{\mathcal{N}} = \{(n, \mathfrak{b}) | n \in \mathcal{N}, n \in \mathfrak{b}\}$ where $\mathfrak{b}$ is a Borel subalgebra. Let $\mathcal{B}$ be the collection of Borel subalgebras in $\mathfrak{g}$. G also acts on $\mathfrak{g}$ by the action Ad. When $G = SL_n$, this is just the conjugation action. In particular Ad permutes Borel subalgebras and thus acts on $\mathcal{B}$ with B the stabilizer of the standard Borel subalgebra. Hence $\mathcal{B} \cong G/B$.

Example 10.2. If $G = SL_n$, then G/B is the variety of full flags in $\mathbb{C}^n$ and the above correspondence sends a flag $0 = F_0 \subset F_1 \subset \cdots F_n = \mathbb{C}^n$ to the set $\mathfrak{b}$ where $x \in \mathfrak{b}$ if $xF_i \subset F_i$.

There is also a natural map to $\mathcal{B}$, the map which forgets the nilpotent element. This is a fibre bundle with fibre $\mathfrak{n}$ and is in fact isomorphic to $T * \mathcal{B} \cong T * G/B$.

Note that $\mathcal{N}$ has a natural orbit stratification via the orbits of the action of G via Ad (in particular, $Ad(g)$ sends nilpotents to nilpotents).

Example 10.3. In the case $G = SL_n$, since the action is conjugation, the orbits are exactly given by the Jordan type of the nilpotent matrix, i.e. they correspond to partitions of n.

Lemma 10.4. *The Springer resolution, $f : \tilde{\mathcal{N}} \to \mathcal{N}$ is semi-small and every stratum is relevant.*

Proof. Our goal is to show that for any $x \in \mathcal{N}$, $\dim_{\mathbb{C}}(f^{-1}(x)) = N - \frac{1}{2}\dim \mathcal{O}_x$ where $\mathcal{O}_x$ is the adjoint orbit of x and $N = \dim_{\mathbb{C}} G/B$ since $2N = \dim_{\mathbb{C}} T^*(G/B) = \dim_{\mathbb{C}} \tilde{\mathcal{N}} = \dim_{\mathbb{C}} \mathcal{N}$.

In particular, note that

$$\dim_{\mathbb{C}}(B) - \dim_{\mathbb{C}}(G_x) = \dim_{\mathbb{C}}(G) - \dim_{\mathbb{C}}(G/B) - \dim_{\mathbb{C}}(G_x) = \dim(\mathcal{O}_x) - N.$$

Here G_x is the stabilizer in G of x.

It turns out that

$$\dim_{\mathbb{C}}(f^{-1}(x)) + \dim_{\mathbb{C}}(B) = \dim_{\mathbb{C}}(\mathcal{O}_x \cap \mathfrak{n}) + \dim_{\mathbb{C}}(G_x).$$

The left hand side is the dimension of $\{g \in G | Ad(g)(\mathfrak{b}) \in f^{-1}(x)\}$, this is the inverse of image of map $g \mapsto Ad(g)(\mathfrak{b})$. If we consider the map $g \mapsto Ad(g^{-1})(x)$ this lies in $\mathcal{O}_x$, and so for the right and side we consider the pullback under the map. Note that $Ad(g^{-1})(x) \in \mathfrak{n}$ iff $Ad(g^{-1})(x) \in \mathfrak{b}$ iff $x \in Ad(g)(\mathfrak{b})$ iff $Ad(g)(\mathfrak{b}) \in f^{-1}(x)$ i.e. both sets have the same dimension.

Combing the above results to obtain the result we want, we just need to show that $\dim_{\mathbb{C}}(f^{-1}(x)) + \dim_{\mathbb{C}}(\mathcal{O}_x \cap \mathfrak{n}) = N$. Now the left hand side can be realized as the dimension of the pullback of $\mathcal{O}_x \cap \mathfrak{n}$ to $\tilde{\mathcal{N}}$, so what is left is to check that this has dimension N. $\square$

Corollary 10.5. *The same method gives a resolution $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$. This map is small.*

Our goal is now analyze $Rf_*\mathbb{C}_{\tilde{\mathcal{N}}}[2N]$. In particular, since this map is semi-small and all strata are relevant, we know that

$$Rf_*\mathbb{C}_{\tilde{\mathcal{N}}}[2N] = \oplus_{(\mathcal{N}_\alpha,\phi)} V_\phi \otimes IC_{\mathcal{N}}(L_\phi),$$

where $\mathcal{N}_\alpha$ varies over relevant strata and ϕ varies over the simple representations of π_1 of $\mathcal{N}_\alpha$. V_ϕ is multiplicity vector space which is the $Hom(H^{BM}_{\text{top}}(f^{-1}(x)), \phi)$ where $x \in \mathcal{N}_\alpha$.

Note that $f^{-1}(x)$ are also known as *Springer fibres.* Our goal is now to show that if ϕ is relevant then V_ϕ is an irreducible representation of W.

Theorem 10.6. *Let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Then if $2n = \dim \mathcal{N}$ then $Hom(Rf_*\mathbb{C}_{\mathcal{N}}[2n], Rf_*\mathbb{C}_{\mathcal{N}}[2n]) \cong H_n^{BM}(Z) \cong C[W]$ where the middle term has an algebra structure given by Borel-Moore convolution. More over $\{Hom(f^{-1}(x), \phi\}$ is a complete collection of irreducible W representations where (x, ϕ) run over G-conjugacy classes of pairs $x \in \mathcal{N}$ and ϕ a relevant irreducible representation of G_x/G_x^o.*

10.1. **Equivariant Decomposition.** In the case that we have a G a linear algebraic group and X an algebraic G-variety, we can think about G-equivariant sheaves on X. Such a sheaf will always be constructible with respect to the orbit stratification. For now assume that $X = G/H$ for some subgroup H and take $x = 1 \cdot H$ as a basepoint. The maps $H \hookrightarrow G \to G/H$ gives rise to a homotopy exact sequence:

$$\cdots \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1.$$

If we work with G connected, then $\pi_0(G) = 1$, so $\pi_1(G/H)$ surjects onto $\pi_0(H) = H/H^o$ where H^o is the identity component. Thus any finite dimensional H/H^o representation is also a $\pi_1(G/H)$ representation.

Lemma 10.7. *A local system $\mathcal{A}$ on G/H is G-equivariant if and only if the corresponding representation of $\pi_1(G/H, x)$ on $\mathcal{A}_x$ is the pullback by the above map of a finite dimensional representation of H/H^o.*

This means that if we have a G-equivariant local system on G/H its decomposition into simple local systems is can be realized by decomposing by the H/H^o representation rather than the $\pi_1(G/H)$ representation on the stalk. This is nicer since for an algebraic group, H/H^o is finite. The upshot of this is that any G-equivariant local system is semisimple as a local system (i.e. forgetting the G-equivariant structure).

Although we could work with a G-equivariant derived category and invoke a G-equivariant version of the decomposition theorem, the definition of such a category is involved. In practice, for $f : M \to N$ a proper semismall map with M non-singular we just want to know how $Rf_*\mathbb{C}_M[\dim_{\mathbb{C}} M]$ decomposes into IC sheaves, equivariant or not. We can how ever exploit the equivariant structure of f and $\mathbb{C}_M$ to derive the following theorem:

Theorem 10.8. *Let M, N be G-varieties such that M is non-singular and N has finitely many G-orbits and $f : M \to N$ be proper, semismall and G-equivariant. If $\mathcal{O}_x$ is the orbit of a point $x \in N$ with $d = \dim_{\mathbb{C}} \mathcal{O}_x$ which is a relevant stratum, then*

$$H^{-d}((Rf_*\mathbb{C}_M[\dim_{\mathbb{C}} M])|_{\mathcal{O}_x}),$$

is a G-equivariant local system.

Proof. Let $\mathcal{A}$ be $Rf_*\mathbb{C}_M[\dim_{\mathbb{C}} M]$. Since $\mathcal{O}_x$ is a stratum on N, we already know that $H^{-d}(\mathcal{A}|_{\mathcal{O}_x})$ is a local system. We just need to check that it is G-equivariant.

Let $\mu : G \times M \to M$ and $\nu : G \times N \to N$ be the action maps and π_M, π_N the projections on the second factor. Then since f is G-equivariant and proper it follows that $\pi_M^* Rf_* = R(\mathrm{id}_G \times f)_* \pi_N^*$ and $\mu^* Rf_* = R(\mathrm{id}_G \times f)_* \nu^*$ by base change.

Since $\pi^* \mathbb{C}_M = \mu^* \mathbb{C}_M = \mathbb{C}_{G \times M}$, the $\mathbb{C}_M$ has a trivial G-equivariant structure α. One can check that by the above reasoning that $R(\mathrm{id}_G \times f)_* \alpha$ satisfies the derived versions of the G-equivariance axioms.

Let $\mathcal{A}$ be the sheaf and denote the above map $\tilde{\alpha} : \pi^* \mathcal{A} \to \nu^* \mathcal{A}$. Then since π^* and ν^* are exact, they commute with homology and we have

$$H^{-d}(\tilde{\alpha}) : H^{-d}(\pi^* \mathcal{A}) = \pi^* H^{-d}(\mathcal{A}) \to H^{-d}(\nu^* \mathcal{A}) = \nu^* H^{-d}(\mathcal{A}).$$

Once again it can be checked that this satisfies the standard G-equivariance axioms. This sheaf is G-equivariant it restriction to an orbit is G-equivariant.

$\square$

It should be noted that in general, the notion of G-equivariance in the derived category is not either of the notions that one might think works. That is, simply asking for the axioms to work in the derived category or taking the derived category of the G-equivariant sheaves is not the correct notion, see [BL].

We can now apply the above lemma to the above theorem and we find that for a relevant stratum $\mathcal{O}_x$ on N, the decomposition of the local system $H^{-d}((Rf_* \mathbb{C}_M[\dim_{\mathbb{C}} M])|_{\mathcal{O}_x})$ via the $\pi_1(\mathcal{O}_x)$ action on the stalk $H^{BM}_{\text{top}}(f^{-1}(x))$ at x is the same as the decomposition via the G_x/G_x^o on the stalk: G_x acts on the fibre $f^{-1}(x)$ and thus permutes the irreducible components of the fibre. The elements of G_x^o must preserve the components, so G_x/G_x^o acts on $H^{BM}_{\text{top}}(f^{-1}(x))$.

Example 10.9. Let us apply this to the case of $f : \tilde{\mathcal{N}} \to \mathcal{N}$ for $G = SL_n$. Both $\tilde{\mathcal{N}}$ and $\mathcal{N}$ are G-varieties and f is G-equivariant. Since $\pi_1(G) = 1$ (SL_n is simply connected), we learn nothing new since this implies that $\pi_1(G/G_x) = G_x/G_x^o$. But, note that f is not defined in terms of SL_n, but its Lie algebra $\mathfrak{sl}_n$. This means that if we consider $G = PGL_n$, the spaces $\tilde{\mathcal{N}}$ and $\mathcal{N}$ are PGL_n-varieties and the map f is PGL_n-equivariant.

The key fact that we can now exploit is that for x nilpotent, $(PGL_n)_x$ is connected. Hence in the decomposition of $Rf_* \mathbb{C}_{\mathcal{N}}$, the only IC sheaves which appear are the constant sheaves on each orbit with multiplicity $H^{BM}_{\text{top}}(f^{-1}(x))$ for some nilpotent x in the orbit.

In this vein, we can say something similar for general G. Associated to a Lie algebra $\mathfrak{g}$, there is the adjoint group G^{ad}, uniquely determined by the fact that it has trivial centre (this is group whose weight lattice is equal it its root lattice). The map $f : \tilde{\mathcal{N}} \to \mathcal{N}$ is G^{ad}-equivariant and hence $Rf_* \mathbb{C}_{\tilde{\mathcal{N}}}$ decomposes into IC sheaves in parallel to the $(G_x^{\text{ad}})/(G_x^{\text{ad}})^o$ decomposition of $H^{BM}_{\text{top}}(f^x)$ over each stratum.

Lemma 10.10. *$(G_x^{ad})/(G_x^{ad})^o$ is abelian if $\mathfrak{g}$ is semisimple.*

Proof. [CM] One checks for the simple lie algebras that $(G_x^{\mathrm{ad}})/(G_x^{\mathrm{ad}})^o$ is a power of $\mathbb{Z}/2\mathbb{Z}$. □

This tells us that the multiplicity spaces for the $(G_x^{\mathrm{ad}})/(G_x^{\mathrm{ad}})^o$ decomposition of $H_{\mathrm{top}}^{BM}(f^x)$ are simply the corresponding isotypic components.

10.2. **The Convolution Algebra.** Let $f_i : M_i \to N$ be proper maps for $i = 1, 2, 3$ with M_i non-singular and $m_i = \dim_{\mathbb{R}} M_i$. Let $Z_{i,j} = M_i \times_N M_j$. Then we have a morphism

$$H_i^{BM}(Z_{1,2}) \times H_j^{BM}(Z_{2,3}) \to H_{i+j-m_2}^{BM}(Z_{1,3}).$$

This map is called Borel-Moore convolution. If $p_{i,j} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ is the projection, then the convolution of two Borel- Moore classes $c_{1,2}$ and $c_{2,3}$ is

$$(p_{1,3})_*((p_{1,2} * c_{1,2}) \cap (p_{2,3} * c_{2,3})).$$

Theorem 10.11. *If $Z = M \times_N M$, then $H_\bullet^{BM}(Z)$ is an algebra and $H(Z) = H_{2\dim_{\mathbb{C}} M}^{BM}(Z)$ is a subalgebra.*

If $f : M \to N$ is semi-small $H(Z)$ is top homology of Z an so has a basis given by the irreducible components of Z.

Corollary 10.12. *There is an $H(Z)$ action on $H_j^{BM}(f^{-1}(x))$.*

Proof. Set $M_1 = M_2 = M$ and $M_3 = \{x\}$ for $x \in N$, then we see that $Z_{2,3} = f^{-1}(x)$ and we have an action of $H(Z)$ on $H_j^{BM}(f^{-1}(x))$. □

Proposition 10.13. *Let $f : M \to N$ be a proper map with M non-singular and $d = \dim_{\mathbb{C}} M$. Then there exists a natural algebra preserving isomorphism*

$$H_\bullet^{BM}(Z) \cong Ext_{D_c^b(N)}(Rf_*\mathbb{C}_M[d], Rf_*\mathbb{C}_M[d]).$$

Proof. We start with the fact that $H_\bullet^{BM}(Z) \cong H^\bullet(Z, \mathbb{D}_Z)$. Then consider the following commuting square:

$$\begin{array}{ccc} Z = M \times_N M & \xrightarrow{\bar{i}} & M \times M \\ \Big\downarrow f_\Delta & & \Big\downarrow f \times f \\ N_\Delta & \xrightarrow{i} & N \times N \end{array}$$

We then have

$$\begin{aligned} H^\bullet(Z, \mathbb{D}_Z) &= H^\bullet(Z, \bar{i}^! \mathbb{C}_{M\times M}) \\ &= H^\bullet(N_\Delta, (f_\Delta)_* \bar{i}^! \mathbb{C}_{M\times M}) \\ &= H^\bullet(N_\Delta, i^!(f\times f)_* \mathbb{C}_{M\times M}) \\ &= H^\bullet(N_\Delta, i^!(Rf_*\mathbb{C}_M \overset{L}{\boxtimes} Rf_*\mathbb{C}_M)) \\ &= H^\bullet(N_\Delta, i^!(Rf_*\mathbb{C}_M \overset{L}{\boxtimes} \mathcal{D}(Rf_*\mathbb{C}_M))) \\ &= H^\bullet(N_\Delta, R\mathcal{H}om^\bullet(Rf_*\mathbb{C}_M, Rf_*\mathbb{C}_M)) \\ &= Ext^\bullet_{D^b_c(N)}(Rf_*\mathbb{C}_M, Rf_*\mathbb{C}_M) \end{aligned}$$

This relies on the following two facts:

$$R\mathcal{H}om(A, B) \cong i^!(B \overset{L}{\boxtimes} DA)$$

and

$$\mathbb{C}_{M\times M} = \mathbb{C}_M \overset{L}{\boxtimes} \mathbb{C}_M.$$

See [CG] theorem 8.6.7 for the proof that this is a morphism of algebras. □

Corollary 10.14. *We have* $H(Z) \cong Hom_{D^b_c(N)}(Rf_*\mathbb{C}_M[d], Rf_*\mathbb{C}_M[d])$.

Corollary 10.15. *If f is semi-small and $Rf_*\mathbb{C}_M[d] = \oplus V_\phi \otimes IC_N(L_\phi)$ where L_ϕ varies over all relevant local systems over all relevant strata N_ϕ, we have $H(Z) \cong \oplus End(V_\phi)$. If χ is the $\pi_1(N_\phi)$ related to L_ϕ, then $V_\phi = Hom_{Rep(\pi_1(N_\phi))}(H^{BM}_{top}(f^{-1}(x)), \chi)$.*

10.3. **Fourier Transform.**

Definition 10.16. Given a variety X with a $\mathbb{C}^*$ action, a sheaf on X is called *monodromic* if it is constructable with respect to the orbits of the $\mathbb{C}^*$ action. A complex of sheaves is monodromic if all its cohomology sheaves are. In particular, $D^b_{\text{mon}}(X)$ is the derived category of such complexes and $Perv_{\text{mon}}(X)$ is the category of perverse monodromic sheaves.

This is typically seen when X is vector space or a vector bundle.

Let I be a complex of injective monodromic sheaves on a vectorspace V. For any open $U \subset V^*$, we construct the *polar set* in V:

$$U^\circ = \{x \in V | Re(f(x)) \geq 0, f \in U\}.$$

We can construct a complex of presheaves associated to I assigning to U the sections of I supported on U°.

The sheafification of this functor, F, is called the Fourier transform. The definition can upgraded to work any vector bundle and sends a complex on that bundle to a complex on the dual bundle. It gives an equivalence of categories $Perv_{\text{mon}}(V)$ and $Perv_{\text{mon}}(V^*)$.

Theorem 10.17. [BR] *The following are the important properties of F:*

1. $F \circ F = \alpha^*$ *where α is the multiplication by -1 function.*
2. *If E is a vector bundle, $i_V : V \to E$ is a subbundle and $\mathbb{C}_V = R(i_V)_*(C)$, then $F(\mathbb{C}_V) = \mathbb{C}_{V^\perp}$ where $V^\perp \subset E^*$ is the annihilator of V.*
3. *For a compact algebraic variety X and vector space V, the following diagram commutes:*

$$\begin{array}{ccc} Perv_{mon}(V \times X) & \xrightarrow{F} & Perv_{mon}(V^* \times X) \\ \downarrow \pi_1^* & & \downarrow \pi_1^* \\ Perv_{mon}(V) & \xrightarrow{F} & Perv_{mon}(V^*) \end{array}$$

10.4. **The geometric construction of $\mathbb{C}[W]$.** Our claim now is that $\mathbb{C}[W] \cong End_{D^b_c(\mathcal{N})}(Rf_*\mathbb{C}_{\mathcal{N}})$.

Since $\mathcal{N} \subset \mathfrak{g}$, we can consider the sheaf $Rf_*\mathbb{C}_{\mathcal{N}}$ as a sheaf on $\mathfrak{g}$, so that instead we can show that $End_{D^b_c(\mathfrak{g})}(Rf_*\mathbb{C}_{\mathcal{N}}) \cong \mathbb{C}[W]$. Note that $\mathbb{C}^*$ acts on $\mathcal{N}$ by multiplication and on $\tilde{\mathcal{N}}$ by just multiplying the nilpotent (i.e. on the fibre of bundle map to $\mathcal{B}$). The Springer resolution f is now $\mathbb{C}^*$-equivariant and thus $Rf_*\mathbb{C}_{\mathcal{N}}$ is monodromic.

Given the above setup, we can now apply the Fourier transform to obtain a a sheaf on $\mathfrak{g}^*$, which has the endomorphisms as $Rf_*\mathbb{C}_{\mathcal{N}}$. We now need to understand what $F(Rf_*\mathbb{C}_{\mathcal{N}})$ is.

Recall that $\tilde{\mathcal{N}} \subset \mathfrak{g} \times \mathcal{B}$ which satisfies the conditions of part b of the theorem 10.17. Thus $\mathbb{C}_{\tilde{\mathcal{N}}}$ considered as a sheaf on $\mathfrak{g}\times\mathcal{B}$ has Fourier transform the annihilator of $\tilde{\mathcal{N}}$ in $\mathfrak{g}^* \times \mathcal{B}$. The fibre of $\tilde{\mathcal{N}}$ above a Borel subalgebra $\mathfrak{b}$ is its nilradical $\mathfrak{n}$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form, then the annihilator of fibre of $\tilde{\mathcal{N}}$ is simply $\mathfrak{b}$, so

$$\tilde{\mathcal{N}}^\perp = \tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} | x \in \mathfrak{b}\}.$$

The Fourier transform of $\mathbb{C}_{\tilde{\mathcal{N}}}$ is the now $\mathbb{C}_{\tilde{\mathfrak{g}}}$. Part c of theorem 10.17 says that $F(Rf_*\tilde{\mathcal{N}}) = R\hat{f}_*\mathbb{C}_{\tilde{\mathfrak{g}}}$ where $\hat{f} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ the Grothendieck-Springer resolution.

Theorem 10.18. *$\hat{f}$ is proper small map.*

The stratification on $\mathfrak{g}$ will have a single open stratum, $\mathfrak{g}^{rs}$, of all regular semisimple elements of $\mathfrak{g}$. Thus

$$R\hat{f}_*\mathbb{C}_{\tilde{\mathfrak{g}}}[\dim_{\mathbb{C}} \tilde{\mathfrak{g}}] = IC_{\mathfrak{g}}(R\hat{f}_*\mathbb{C}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g}^{rs}}).$$

Finally, we apply the *perverse continuation principle*, which says that any morphisms between IC sheaves is a unique extension of a morphism between the local systems which define them. Thus we now have reduced the problem to showing that $\mathbb{C}[W] \cong End(R\hat{f}_*\mathbb{C}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g}^{rs}})$.

Let $\tilde{\mathfrak{g}}^{rs} = \hat{f}^{-1}(\mathfrak{g}^{rs})$, then

Lemma 10.19. (1) *There is a free W action on $\tilde{\mathfrak{g}}^{rs}$ such that $\tilde{\mathfrak{g}}^{rs}/W \cong \mathfrak{g}^{rs}$ and the quotient map is $\hat{f}$.*
(2) *The map $\hat{f}$ is a regular covering with automorphism group W.*

Since the covering is regular with automorphism group W, it follows that $\pi_1(\mathfrak{g}^{rs})$ has W as a quotient and in particular the monodromy action of on the fibre of $\hat{f}$ over a regular semi-simple element x comes from the natural W action. But this fibre is itself nothing but W, and the W representation $H^{BM}_{\text{top}}(f^{-1}(x)) \cong C[W]$.

As the regular representation we see that

$$C[W] = \oplus_\gamma L_\gamma \otimes V_\gamma = \oplus End(L_\gamma),$$

where V_γ runs over the irreducible W representations and $L_\gamma = (V_\gamma)^*$ are the multiplicity spaces. Note that this isomorphism is also a isomorphism of algebras and this also the decomposition under the monodromy action.

Thus

$$R\hat{f}_*\mathbb{C}_{\tilde{\mathfrak{g}}}[\dim_{\mathbb{C}} \tilde{\mathfrak{g}}] = \oplus L_\gamma \otimes IC_{\mathfrak{g}}(A_\gamma).$$

11. Geometric Satake Correspondence

In this section, we consider G a complex, connected reductive algebraic group. By $\mathcal{O}$ we denote the ring $\mathbb{C}[[t]]$ of formal power series and by $\mathcal{K}$ its fraction field $\mathbb{C}((T))$ of Laurent series. $G(\mathcal{K})$ and $G(\mathcal{O})$ will denote the $\mathcal{K}$ and $\mathcal{O}$ points of G respectively.

Definition 11.1. The *Affine Grassmannian* for G is $\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$.

It is an infinite dimensional space and not in general a group. In particular it is an example of an *ind-variety* or *ind-scheme*, a direct limit of a family of varieties by closed embeddings.

Example 11.2. (1) Suppose that G is a torus, then Gr is just $X_*(T) = Hom(\mathbb{C}^\times, T)$ the coweights of T.
(2) Our main example will be $G = PGL_n$, in which case the elements of $G(\mathcal{K})$ are n by n matrices in $\mathcal{K}$ with non 0 determinant, up to multiplication by an element of $\mathcal{K}^\times$. The elements of $G(\mathcal{O})$ are are n by n matrices in $\mathcal{O}$ with determinant in $\mathcal{O}^\times$ up to multiplication by an element of $\mathcal{O}^\times$. This is equivalent to asking that setting $z = 0$ leaves one with an invertible matrix in $G(\mathbb{C})$.

Note that the columns of any representative of an element in Gr give a basis for a rank n $\mathcal{O}$ lattice in $\mathcal{K}^n$. The fact that we work with PGL_n means that the lattice is only determined up to multiplying the lattice by t^k for $k \in \mathbb{Z}$.

$G(\mathcal{O})$ acts on Gr by left multiplication and if $T \subset G$ is a maximal torus with W the Weyl group, then the above orbits are parametrized by the W orbits on $X_*(T)$, i.e. once we fix a Borel B containing T, by the dominant coweights of G. Implicitly, a coweight $\lambda \in X_*(T)$ can be considered as a function $\lambda : \mathbb{C}^\times \to T \subset G$, since this function is algebraic it is an element of $G(\mathcal{K})$. We will denote its image in Gr by t^λ. Let $\mathrm{Gr}^\lambda_\circ = G(\mathcal{O})t^\lambda$, then we have

Lemma 11.3. $\mathrm{Gr}^\lambda = \overline{\mathrm{Gr}^\lambda_\circ} = \bigcup \mu \leq \lambda \mathrm{Gr}^\lambda_\circ$ *where* λ *and* μ *are dominant coweights.*

Example 11.4. In the case of PGL_n, a coweight λ is determined by a sequence of n integers λ_i, up to adding the all ones sequence. The actual map sends $t \in \mathbb{C}$ to the diagonal matrix $\mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n})$. Thus $t^\lambda = \mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n})G(\mathcal{O})$. Note that this does not depend on the representative since G is PGL_n and $t^{(1,\ldots,1)} \in Z(GL_n(\mathcal{K}))$.

We will use t^0 to denote the identity matrix. Note that if we consider the action of G on t^λ, the stabilizer is the parabolic subgroup P_λ associated to λ. Hence $Gt^\lambda = G/P_\lambda \subset \mathrm{Gr}^\lambda_\circ$. In particular, Gr^λ is a G-equivariant bundle over Gt^λ.

Lemma 11.5. *There is a natural* $\mathbb{C}^\times$ *action* Gr*, where* $s \in \mathbb{C}^\times$ *acts by multiplying* t *by* s*. This is called loop rotation.*

$$\mathrm{Gr}^\lambda_\circ = \{x \in \mathrm{Gr} | \lim_{s \in \mathbb{C}^\times | s \to 0} sx \in Gt^\lambda\}.$$

This also shows that $\mathrm{Gr}^\lambda_\circ$ are simply connected. We can also determine that $\dim_{\mathbb{C}} \mathrm{Gr}^\lambda_\circ = 2\rho(\lambda)$ where ρ is half the sum of the positive roots of G.

The map $ev_0 : G(\mathcal{O}) \to G$ is the evaluation at 0 map, the Iwahori subgroup $I = ev_0^{-1}(B)$. Where the $G(\mathcal{O})$ orbits are parametrized by dominant coweights, the Iwahori orbits are parametrized by coweights. Let $K = ev_0^{-1}(1)$, then $Kt^\lambda \subset \mathrm{Gr}^\lambda_\circ$ is the typical fibre of the vector bundle $\mathrm{Gr}^\lambda_\circ \to Gt^\lambda$.

We now see that Gr is ind-variety, if Gr_i is the union of the $G(\mathcal{O})$ orbits of dimension at most i, then each Gr_i is closed and there is a closed embedding of $\mathrm{Gr}_i \hookrightarrow \mathrm{Gr}_{i+1}$. Each of these has the structure of a variety and since Gr is the direct limit of this system, it is an ind-variety. Each Gr_i is $G(\mathcal{O})$ stable and thus has a stratification by the $\mathrm{Gr}(\mathcal{O})$ orbits that form it. We define $Perv(\mathrm{Gr})$ as the direct limit of the categories $Perv(\mathrm{Gr}_i)$.

Our goal is to show:

Theorem 11.6.

$$Perv(\mathrm{Gr}) \cong Rep(\hat{G}).$$

where $\hat{G}$ is the Langlands dual of G.

Definition 11.7. Given a complex reductive connected algebraic group G with root datum $(X^*, \Lambda, X_*, \Lambda^\vee)$, the Langlands dual $\hat{G}$ has root datum $(X_*, \Lambda^\vee, X^*, \Lambda)$.

Example 11.8. (1) For SL_n, the weight lattice is $\mathbb{Z}^n/(1, \ldots, 1)$ and the root lattice are those weights with a representative that sums to 0, (i.e. $(1,-1,0,\dots,0)$). The coweight and coroot lattices are the same and are $\{(a_1,\dots,a_n)\in\mathbb{Z}^n | a_1+\dots+a_n=0\}$. Swapping these lattice pairs, the associated group is PGL_n.

(2) For GL_n, the weight and coweight lattices are just $\mathbb{Z}^n$ and the root and coroot lattices are $\{(a_1,\dots,a_n)\in\mathbb{Z}^n | a_1+\dots+a_n=0\}$. Thus the Langlands dual of GL_n is GL_n.

(3) $\hat{SO}_{2n+1} = Sp_{2n}$

(4) $\hat{SO}_{2n} = SO_{2n}$

The proof of the above result will essentially take three steps: first we will describe a non-standard monoidal structure on $Perv(\mathrm{Gr})$. Recall that as part of the monoidal structure we have an object $\not\Vdash$ which is the identity object. Then we will show that $Perv(\mathrm{Gr})$ is a *neutral Tannakian category*, that is, it is a rigid abelian monoidal category with a monoidal functor (typically called the *fibre functor*) to $Vect_k$ that is exact and faithful where $k = End(\not\Vdash)$.

In this case we will use Tannaka-Krein duality (see [DM] for more information):

Theorem 11.9. *Given a neutral Tannakian category C with fibre functor F, the group of natural transformations $Aut^{\otimes}(F)$ of F to itself respecting the monoidal structure is an affine group scheme (in particular it is a inverse limit of algebraic groups). In particular each $F(A)$ for $A \in C$ has the structure of an $Aut^{\otimes}(F)$ representation and thus F is functor from C to $Rep_k(Aut^{\otimes}(F))$ ($k = End(\not\Vdash)$) which is an equivalence of categories.*

Thus we will find that $Perv(\mathrm{Gr})$ is the representation category of some algebraic group and will show that the root datum of this group is dual to the root datum of G.

It should be noted that Gr is often not connected, i.e. when $G = GL_n$ it has $\mathbb{Z}$ many components: the orbit of coweight λ lies in component $|\lambda|$. For $G = PGL_n$, there are n components with the orbit of coweight λ in component $|\lambda| \mod n$.

11.1. **Monoidal Structure on** $Perv(Gr)$**.** To begin with it should be noted that the the category $Perv(\mathrm{Gr})$ already has a set of simple objects in bijection with the irreducible representations of $\hat{G}$. Since each stratum is simply connected and corresponds to dominant coweight of G, we should think of the perverse sheaf $IC_\lambda = IC(\mathbb{C}_{\mathrm{Gr}^\lambda_\circ})$ as corresponding to the irreducible representation V_λ of $\hat{G}$ of highest weight λ. Since we know that the representation $V_{\lambda+\mu}$ will always be a subrepresentation of the representation $V_\lambda \otimes V_\mu$, we want $IC_{\lambda+\mu}$ to be a subobject of $IC_\lambda \otimes IC_\mu$. If we are using the standard tensor product this will not happen since the support of the tensor product will not in general contain $\mathrm{Gr}^{\lambda+\mu}$.

This means that we need to define a different monoidal structure on $Perv(\mathrm{Gr})$ if we want the above identification to give an equivalence of categories.

Rather than dealing with $G(\mathcal{O})$ and $G(\mathcal{K})$, it is also possible to use $L^+G = G(\mathbb{C}[t])$ and $LG = G(\mathbb{C}[t,t^{-1}])$. Since $L^+G \subset G(\mathcal{O})$ and $LG \subset G(\mathcal{K})$, there exists a map $j : LG/L^+G \to \mathrm{Gr}$. This map is injective since $L^+G = LG \cap G(\mathcal{O})$.

Proposition 11.10. [G95] *The map j above is an LG equivariant isomorphism. Each $G(\mathcal{O})$ orbit in* Gr *is the image of a single L^+G orbit on LG/L^+G.*

Consider Ω, the set of based polynomial loops in the maximal compact subgroup K, i.e. polynomial maps f from S^1 to K such that $f(1) = 1$. Considering S^1 as the unit circle in $\mathbb{C}$, we can extend any such map to $f \in LG$, so $\Omega \subset LG$. There exists an 'Iwasawa' decomposition of LG: any $l \in LG$ can be decomposed uniquely as a product of an element in Ω and an element of LG that can be extended to 0 i.e. an element of L^+G.

This means that $\mathrm{Gr} \cong \Omega$ and thus obtains a group structure. A word of caution here, since $G(\mathcal{O})$ is not normal in $G(\mathcal{K})$, the multiplication of cosets in Gr is not well defined. What we have done is to select in each coset a unique element which comes from Ω. The multiplication structure is then give by the multiplication structure from Ω.

Definition 11.11. The *convolution product* of $L, M \in D^b_c(Gr)$:

$$L \star M = Rm_*(L \overset{L}{\boxtimes} M)$$

where m is the multiplication map above.

Theorem 11.12. [G95]

(1) *If $L, M \in Perv(Gr)$, then $L \star M \in Perv(Gr)$*
(2) *This makes $Perv(Gr)$ into a rigid monoidal abelian category.*
(3) *The functor $H^\bullet : Perv(Gr) \to Vect$ is exact, faithful and monoidal.*

It is possible to define the convolution product of two simple objects in this category as the pushfoward via a semi-small map of an IC sheaf. Let λ, μ be dominant coweights of G. Let $a, b \in \mathrm{Gr}$ and $\hat{a}$ and $\hat{b}$ representatives of a and b respectively. Then then $\hat{a^{-1}}\hat{b}G(\mathcal{O})$ is not a well defined coset, its orbit is well defined. This means that we have function $d(a,b)$ which takes values in coweights of G.

One can check that $\mathrm{Gr}^\lambda_\circ = \{a \in \mathrm{Gr} | d(t^0, a) = \lambda\}$ and $\mathrm{Gr}^\lambda = \{a \in \mathrm{Gr} | d(t^0, a) \leq \lambda\}$.

Definition 11.13. The twisted product

$$\mathrm{Gr}^\lambda \tilde{\times} \mathrm{Gr}^\mu = \{(L_0 = t^0, L_1, L_2) | d(L_0, L_1) \leq \lambda, d(L_1, L_2) \leq \mu\}.$$

This space is naturally stratified by (ν, γ) with $\nu \leq \lambda$ and $\gamma \leq \mu$:

$$\mathrm{Gr}^\nu_\circ \tilde{\times} \mathrm{Gr}^\gamma_\circ = \{(L_0 = t^0, L_1, L_2) | d(L_0, L_1) = \nu, d(L_1, L_2) = \gamma\}.$$

Proposition 11.14.

$$IC_\lambda \star IC_\mu \cong R\pi_* IC_{\lambda,\mu}$$

where

$$IC_{\lambda,\mu} = IC(\mathbb{C}_{\mathrm{Gr}_\circ^\lambda \tilde{\times} \mathrm{Gr}_\circ^\mu})$$

and where π is the projection on the last factor.

Before proving this, we must first understand that $IC_\lambda \overset{L}{\boxtimes} IC_\mu$ is isomorphic to $IC(\mathbb{C}_{\mathrm{Gr}_\circ^\lambda \times \mathrm{Gr}_\circ^\mu})$. This will be the case if the external product satisfies the same support conditions, is self Verdier dual and has the same stalks in the $-\rho(\lambda+\mu)$ cohomology over $\mathrm{Gr}_\circ^\lambda \times \mathrm{Gr}_\circ^\mu$.

We will need to use the following 3 lemmas:

Lemma 11.15. [?] *Let $A, C \in D_c^b(X)$ and $B, D \in D_c^b(Y)$ for some stratified spaces X and Y, then there is an isomorphism:*

$$R\mathcal{H}om(A,C) \overset{L}{\boxtimes} R\mathcal{H}om(B,D) \cong R\mathcal{H}om(A \overset{L}{\boxtimes} B, C \overset{L}{\boxtimes} D).$$

Lemma 11.16. *Let $f : Y \to X$ and $A, B \in D_c^b(X)$, then $f^*(A \overset{L}{\otimes} B) = (f^*A) \overset{L}{\otimes} (f^*B)$.*

Proof. This is true on the level of sheaves and since $\otimes$ and f^* are exact functors, it will be true at the level of the derived category. The only possible problem is that $A \overset{L}{\otimes} B$ is the total complex of the bicomplex $A^i \otimes B^j$, but the functor f^* is additive. □

Lemma 11.17. *Let $A, B \in D_c^b(X)$, then*

$$H^i(A \overset{L}{\otimes} B) = \oplus_{p+q=i} H^p(A) \otimes H^q B.$$

This is a consequence of the fact that $\otimes$ is an exact bifunctor at the level of $Sh(X)$ since our sheaves are sheaves of vectorspaces.

Theorem 11.18. *Let $X = \cup X_n$ and $Y = \cup Y_n$ be stratified varieties with the given strata. Let $\dim_{\mathbb{C}} X = d_x$, $\dim_{\mathbb{C}} Y = d_y$, $\dim_{\mathbb{C}} X_n = n$ and $\dim_{\mathbb{C}} Y_n = n$. Then*

$$IC(\mathbb{C}_{X_{d_x}}) \overset{L}{\boxtimes} IC(\mathbb{C}_{Y_{d_y}}) \cong IC(\mathbb{C}_{X_{d_x} \times Y_{d_y}}).$$

Proof. Let $A = IC(\mathbb{C}_{X_{d_x}})$ and $B = IC(\mathbb{C}_{Y_{d_y}})$. Applying lemma 11.15 with $C = \mathbb{D}_X$ and $D = \mathbb{D}_Y$ and noting that $\mathbb{D}_X \overset{L}{\boxtimes} \mathbb{D}_Y \cong \mathbb{D}_{X\times Y}$ gives $\mathcal{D}(A \overset{L}{\boxtimes} B) \cong (\mathcal{D}A) \overset{L}{\boxtimes} (\mathcal{D}B) \cong A \overset{L}{\boxtimes} B$, since A and B are Verdier self dual.

Applying the other two lemmas where $f : (a,b) \hookrightarrow X \times Y$, gives

$$H^i((A \overset{L}{\boxtimes} B)_{(a,b)}) = \oplus_{p+q=i} H^p(A_a) \otimes H^q(B_b).$$

Recall that $H^p(A_a) = 0$ if $p < -d_x$ and $p >= -n$ where $a \in X_n$ unless $n = d_x$, whence $H^p(A_a) = 0$ if $p > -d_x$. In fact if $a \in X_{d_x}$ then $H^p(A_a)$

is 0 unless $p = -d_x$, in which case it is $\mathbb{C}$. We have a similar statement for $H^q(B_b)$.

Let $(a,b) \in X_{d_x} \times Y_{d_y}$, then $H^i((A \overset{L}{\boxtimes} B)_{(a,b)}) = \oplus_{p+q=i} H^p(A_a) \otimes H^q(B_b)$. But for both $H^p(A_a)$ and $H^q(B_b)$ to not be trivial, we must have $p = -d_x$ and $q = -d_y$. Hence over the open stratum, we only have support in degree $-d_x - d_y$ as needed.

Let $(a,b) \in X_j \times Y_k$ where $(j,k) \neq (d_x, d_y)$. Then let $i < -d_x - d_y$, then if $p+q = i$, we cannot have $p \geq -d_x$ and $q \geq -d_y$, so in $H^i((A \overset{L}{\boxtimes} B)_{(a,b)}) = \oplus_{p+q=i} H^p(A_a) \otimes H^q(B_b)$ at least one term in each tensor product is trivial.

If $i \geq -j-k$, wlog let us assume that if one of $j = d_x$ or $k - d_y$, that it is the first. Then for $H^p(A_a) \otimes H^q(B_b)$ to be non trivial we must have $-d_x \leq p \leq -j$ and $-d_y \leq q \leq -k-1$ (we have $-j$ rather than $-j-1$ since we may have $j = d_x$). But then $p+q \leq -j-k-1 < i$.

Thus $A \overset{L}{\boxtimes} B$ satisfied the support conditions of IC sheaf arising from a local system on $X_{d_x} \times Y_{d_y}$. Which local system is it? It it the local system given by the restriction of $H^{-d_x-d_y}(A \overset{L}{\boxtimes} B)$ to $X_{d_x} \times Y_{d_y}$. Once again, by the above lemmas this is

$$\oplus_{p+q=-d_x-d_y} H^p(A|_{X_{d_x}}) \otimes H^q(B|_{Y_{d_y}}) = H^{-d_x}(A|_{X_{d_x}}) \otimes H^{-d_y}(B|_{Y_{d_y}}) = \mathbb{C}_{X_{d_x}} \otimes \mathbb{C}_{Y_{d_y}} = \mathbb{C}_{X_{d_x} \times Y_{d_y}}.$$

$\square$

Proof of proposition 11.14. We identify Gr and Ω, we have a commuting triangle

$$\begin{tikzcd} \Omega \times \Omega \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \Omega \times \Omega \arrow[dl, "m"] \\ & \Omega & \end{tikzcd}$$

Here $f(x_1, x_2) = (x_1, x_1^{-1} x_2)$.

$$IC_\lambda \star IC_\mu = Rm_*(IC_\lambda \overset{L}{\boxtimes} IC_\mu) = R\pi_* f^*(IC_\lambda \overset{L}{\boxtimes} IC_\mu).$$

Note that if we make the triangle above a square by adding an identity morphism to the bottom, then the result is isomorphic to the pullback square for the identity and m, hence we would like to apply base change which would result in the above formula. But to do this m must be proper.

We can be sure this is true, since $IC_\lambda \overset{L}{\boxtimes} IC_\mu$ is supported on $\mathrm{Gr}^\lambda \times \mathrm{Gr}^\mu$ (as shown above), we can replace the left term in the above triangle with $\mathrm{Gr}^\lambda \tilde{\times} \mathrm{Gr}^\mu$, the right with $\mathrm{Gr}^\lambda \times \mathrm{Gr}^\mu$ and the bottom with $\mathrm{Gr}^{\lambda+\mu}$, all of which are compact spaces. The result is still a pullback square but now the left and right sides are proper. Note that in to even make sense of operations like

f^*, one needs to restrict to some finite dimensional strata where we know things will work by the standard theory.

To prove the result, if $f^*(IC_\lambda \boxtimes IC_\mu) \cong IC_{\lambda,\mu}$, then we would be done. But f is an isomorphism which sends stratum $\mathrm{Gr}^\nu_\circ \tilde{\times} \mathrm{Gr}^\gamma_\circ$ of $\mathrm{Gr}^\lambda \tilde{\times} \mathrm{Gr}^\mu$ to $\mathrm{Gr}^\nu_\circ \times \mathrm{Gr}^\gamma_\circ$. Thus f^* will send the sheaf $IC_\lambda \overset{L}{\boxtimes} IC_\mu \cong IC(\mathbb{C}_{\mathrm{Gr}^\lambda_\circ \times \mathrm{Gr}^\mu_\circ})$ to $IC_{\lambda,\mu}$. □

Lemma 11.19. *The map π above is proper semi-small.*

Proof. The image of the map π is $\mathrm{Gr}^{\lambda+\mu}$, which is stratified by orbits $\mathrm{Gr}^\nu_\circ$ with $\nu \leq \lambda + \mu$. To show π is semi-small we need to check that

$$\dim_{\mathbb{C}} \pi^{-1}(t^{\omega_0 \nu}) \leq \frac{1}{2}(\dim_{\mathbb{C}} \mathrm{Gr}^{\lambda+\mu} - \dim_{\mathbb{C}} \mathrm{Gr}^\nu_\circ).$$

The right hand side simplifies to $\rho(\lambda + \mu - \nu)$. We are free to check the inequality at any point in the orbit $\mathrm{Gr}^\nu_\circ$ since all the fibres are related by $G(\mathcal{O})$. The choice of $t^{\omega_0\nu}$ is for ease of calculation. The fibre is made of those points in Gr^λ whose distance to $t^{\omega_0\nu}$ is at most μ.

We will calculate this dimension by using the next few results. □

11.1.1. *Semi-infinite orbits.* Let N be the unipotent radical of B, (i.e. in the case of SL_n this would be the upper triangular matrices with a diagonal of 1). We can consider $N(\mathcal{K}) \subset G(\mathcal{K})$, the $N(\mathcal{K})$ orbits on Gr are parametrized by coweights of G. We set $S^\circ_\nu = N(\mathcal{K})t^\nu$ for any coweight of G. These are called the *semi-infinite* orbits, since they have neither finite dimension or finite codimension.

We have $S_\nu = \overline{S}^\circ_\nu = \cup_{\mu \leq \nu} S^\circ_\mu$, see [MV] (Prop 3.1) for a proof.

Mirković and Vilonen use the seminfinite orbits to prove a variety of interesting results. Of particular interest to them is the intersection of $S_\nu \cap \mathrm{Gr}^\lambda$:

Theorem 11.20. *$S_\nu \cap \mathrm{Gr}^\lambda_\circ$ is nonempty precisely when $t^\nu \in \mathrm{Gr}^\lambda$. $S_\nu \cap \mathrm{Gr}^\lambda$ is of pure dimension $\rho(\nu + \lambda)$ when λ is dominant.*

Proof. Let $\rho^\vee$ be half the sum of the positive coroots, then under the conjugation action $\rho^\vee(s)n$ for $n \in N$, we see that $\lim_{s\to 0} \rho^\vee(s)n = 1$. Thus we have $\lim_{s\to 0} \rho^\vee(s)x = t^\nu$ for any $x \in S_\nu$ and since the t^ν are the fixed points for this $\rho^\vee$ action, it follows that

$$S_\nu = \{x \in \mathrm{Gr} | \lim_{s\to 0} \rho^\vee(s)x = t^\nu\}.$$

If $x \in S_\nu \cap \mathrm{Gr}^\lambda_\circ$, then because $\mathrm{Gr}^\lambda_\circ$ is T-invariant, we see that $t^\nu \in \mathrm{Gr}^\lambda$. In particular, we can now assume that ν is a weight in the irreducible representation of weight λ for $G^\vee$.

First we claim that $S_\nu \cap \mathrm{Gr}^\nu = N(\mathcal{O})t^\nu$. If K_- is the kernel of the evaluation at ∞ map applied to $G(\mathbb{C}[z^{-1}])$, then $K_- t^\nu$ is the fibre of the opposite orbit $\mathrm{Gr}^\circ_\lambda$ over t^ν.

$$S_\nu \cap \mathrm{Gr}^\nu = N(\mathcal{O})(N(\mathcal{K}) \cap K_-)t^\nu \cap \mathrm{Gr}^\nu = N(\mathcal{O})\left((N(\mathcal{K}) \cap K_-)t^\nu \cap \mathrm{Gr}^\nu\right).$$

$$(N(\mathcal{K}) \cap K_-)t^\nu \subset K_- t^\nu \subset \mathrm{Gr}^\circ_\nu.$$

But the intersection $\mathrm{Gr}^\circ_\nu \cap \mathrm{Gr}^\nu = Gt^\nu$, so the intersection $(N(\mathcal{K}) \cap K_-)t^\nu \cap \mathrm{Gr}^\nu = t^\nu$.

If ν is anti-dominant, $N(\mathcal{O})$ stabilizes t^ν, so the result is the single point t^ν, otherwise if ν is dominant, then $N^-(\mathcal{O})$ (the unipotent of the opposite Borel), stabilizes t^ν and $It^\nu = B(\mathcal{O})N^-(\mathcal{O})t^\nu = B(\mathcal{O})t^\nu = N(\mathcal{O})t^\nu$ where I is the Iwahori.

In particular, this prove the theorem when ν is λ or $\nu = w_0\lambda$ where w_0 is the longest word in W.

For the details of the remainder of the proof, check [MV] (Theorem 3.2). One first shows that the boundary of S_ν in $\overline{S}_\nu$ is given by a hyperplane section. In particular, if we start with a dimension d irreducible component C of $S_\nu \cap \mathrm{Gr}^\lambda$ and then select a component of the intersection of C with the hyperplane for S_ν, the result is a component of dimension $d-1$. This component has dense intersects another of the S_μ with $\mu < \nu$. We apply the process again, until we are left with a component of dimension 0 and a strictly decreasing chain of d coweights between ν and the anti-dominant $\omega_0\lambda$. Since such a chain as length at most $\rho(\nu - \omega_0\lambda)$, it follows that $d \leq \rho(\nu - \omega_0\lambda)$.

Similarly working from a component of $S_\lambda \cap \mathrm{Gr}^\lambda$ which contains C, we obtain a sequence of weights whose length is the codimension of C in Gr^λ. This chain's length will be bounded by $\rho(\lambda - \nu)$.

These two conditions together with the fact that $\dim_{\mathbb{C}} C + \mathrm{codim}_{\mathbb{C}} = 2\rho(\lambda)$, give the desired result. □

Corollary 11.21. *For any dominant coweight λ, and any T invariant closed subset $X \subset \mathrm{Gr}^\lambda$, we have* $\dim_{\mathbb{C}}(X) \leq \max_{t^\nu \in X^T} \rho(\lambda + \nu)$.

Proof. $X \cap S_\nu$ is non-empty precisely when $t^\nu \in X$. Since

$$X = \cup_{t^\nu \in X^T} X \cap S_\nu \subset \cup_{t^\nu \in X^T} \mathrm{Gr}^\lambda \cap S_\nu.$$

□

Continuation of lemma 11.19. Recall that we must compute dimension of the variety $X \subset \mathrm{Gr}^\lambda$ which consists of those points $xG(\mathcal{O}) \in \mathrm{Gr}^\lambda$ such that $d(xG(\mathcal{O}), t^{\omega_0\nu}) \leq \mu$ and show that it is bounded above by $\rho(\lambda + \mu - \nu)$. The variety X is T invariant: we know that $x^{-1}t^{\omega_0\nu} \in \mathrm{Gr}^\mu$, hence $x^{-1}h^{-1}t^{\omega_0\nu} \in \mathrm{Gr}^\mu$ since $t^{\omega_0\nu}$ is a T fixed point. Thus $d(hxG(\mathcal{O}), t^{\omega_0\nu}) \leq \mu$.

Thus $\dim_{\mathbb{C}} X \leq \max_{t^\gamma \in X^T} \rho(\lambda + \gamma)$. Note that we must have $d(t^\gamma, t^{\omega_0\nu}) \leq \mu$, in particular we have $t^{\omega_0\nu - \gamma} \in Gr^\mu$, thus it follows that $\omega_0\nu - \gamma$ is a weight of the representation of $G^\vee$ of highest weight μ. Thus $\omega_0\nu - \gamma + \mu$ is a positive and so $0 \leq \rho(\omega_0\nu - \gamma + \mu) = \rho(-\nu - \gamma + \mu)$. Thus

$$\rho(\lambda + \gamma) \leq \rho(\lambda + \gamma) + \rho(-\nu - \gamma + \mu) = \rho(\lambda + \mu - \nu).$$

□

At this point these is some concern the existence of an isomorphism $A \star B \cong B \star A$. This and the fact that $H^*(\mathrm{Gr}, \cdot)$ is a tensor functor with respect to this isomorphism requires a slightly different view point of the product $\star$ and the Grassmannian, see sections 5 and 6 of [MV].

In [G], a slightly different approach is taken. We fix a Cartan anti-involution on G, this is a map θ which fixes a maximal torus T such that $T \cap K$ is maximal in the maximal compact K and sends the Borel B to its opposite Borel. For example, on PGL_n one such map is the transposition map. If $n, m \in K$ then $\theta(nm) = \theta(m)\theta(n)$. The anti-involution preserves Ω and more over will preserve L^+G as well. Since it fixes T we have $\theta(t^\lambda) = t^\lambda$ and thus we see that $\theta^* IC_\lambda \cong IC_\lambda$, since it preserves the orbit structure. Recall that $Perv(\mathrm{Gr})$ is semi-simple so the same is true for any $A \in Perv(\mathrm{Gr})$, although not canonically. Together, this gives a chain of isomorphisms:

$$A \star B \cong \theta(A) \star \theta(B) \cong \theta(B \star A) \cong B \star A.$$

11.2. **Weight Functors.**

Definition 11.22. For each coweight ν, the functor $F_\nu : Perv(\mathrm{Gr}) \to Vect$ defined by $F_\nu(A) = H_c^{2\rho(\nu)}(S_\nu, A)$ is the ν *weight functor*.

These functors will be an important tool in studying the group $\tilde{G}$ that arises from the Tannakian formalism.

Theorem 11.23 ([MV] Thm. 3.5)**.** *F_ν are exact.*

Theorem 11.24 ([MV] Thm. 3.6)**.**

$$H^*(\mathrm{Gr}, \cdot) \cong F = \oplus_\nu F_\nu.$$

Theorem 11.25. [MV] *$A \in D_c^b(\mathrm{Gr})$ is perverse if and only if $H^i(S_\nu, A) = 0$ for all $i \neq 2\rho(\nu)$.*

From the above two results, we conclude that the fibre functor H^* is exact. Since it is exact, the fact that it is faithful follows immediately if for any perverse sheaf A there exists a coweight ν such that $F_\nu(A) \neq 0$. This shows that H^* does not annihilate any non-zero objects.

Theorem 11.26.

$$F_\nu(IC_\lambda) = \mathbb{C}[Irr(\mathrm{Gr}^\lambda \cap S_\nu)].$$

These components are the so called 'MV'-cycles. The proof follows from deducing that $H_c^{2\rho(\nu)}(S_\nu \cap \mathrm{Gr}^\lambda, IC_\lambda) \cong H_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathrm{Gr}^\lambda, \mathbb{C})$.

11.3. **The Theorem.** One now has to verify that $Perv(G)$ is a Neutral Tannakian category with fibre functor H^*. Once this is done, it follows that there is an affine group scheme $\tilde{G}$ such that $Perv(G) \cong Rep(\tilde{G})$. We want to check that $\tilde{G} \cong G^\vee$. First we must determine why $\tilde{G}$ is an algebraic group.

Theorem 11.27. *An affine group scheme $\tilde{G}$ is algebraic iff it has a faithful finite dimensional representation.*

Now consider $M = \oplus_i IC_{\omega_i}$ where ω_i are the fundamental coweights of G (in general one can take any finite set of dominant coweights that positively generate the set of dominant coweights). Then any simple object appears in $M^{\star j}$ for some j: Take a coweight $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ and set $j = a_1 + \cdots + a_n$. Then the product $IC_{\omega_1}^{\star a_1} \star \cdots \star IC_{\omega_n}^{\star a_n}$ is a summand of $M^{\star j}$ and the multiplication map is one to one over the orbit $\mathrm{Gr}_\circ^\lambda$, so IC_λ occurs (with multiplicity 1) in the decomposition. It now follows that $H^*(M \star M^*)$ is a faithful representation of $\tilde{G}$.

The semisimplicity of $Perv(G)$ shows that $\tilde{G}$ is reductive. Thus its is identified by its root system. The final conclusion follows by arguing that since H^* factors through the category of coweight graded vector spaces, the dual torus to $T^\vee$ to T lies inside $\tilde{G}$. For each $a \in T^\vee$ one constructs a natural transformation that respects the tensor structure, thus embedding $T^\vee \subset \tilde{G}$. The weights ν for the irreducible representation V_λ of $G^\vee$ are thus the same as the $T^\vee$ weights of $F(IC_\lambda)$. We can then immediately conclude that $\tilde{G} \cong G^\vee$.

12. Kazhdan-Lusztig polynomials

We are following the note of Simon Riche, entitled *Perverse sheaves on flag manifolds and Kazhdan-Lusztig polynomials.*

12.1. **Hecke algebra.** For W the Weyl group of G, complex connected semisimple algebraic group, we can associate the Hecke algebra $\mathcal{H}_W$. It is a $\mathbb{Z}[t, t^{-1}]$ algebra which is a deformation of W. It is generated (as an algebra) by T_s where s a simple reflection in W. We impose the usual braid condition from W on T_s and T_s' when $s \neq s'$. Where $s^2 = 1$ in W, we have $T_s^2 = (t^2-1)T_s + t^2$. Note that setting $t = 1$ gives back the standard relation for W.

Lemma 12.1. *For $w \in W$, there exists a well defined element T_w. Given a simple reflection s, if $l(sw) > l(w)$, then we $T_sT_w = T_{sw}$. If $l(sw) < l(w)$, then we have $T_sT_w = (t^2-1)T_w + t^2T_{sw}$. If $w, v \in W$ with $l(w)+l(v) = l(wv)$ then $T_vT_w = T_{vw}$.*

Proof. If $w \in W$ with reduced expression $w = s_{i_1} \cdots s_{i_n}$ then define $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$. This is well defined since any two reduced words are related by a sequence of braid moves. If $l(sw) = 1 + l(w)$, then $ss_{i_1} \cdots s_{i_n}$ is a reduced word and $T_sT_w = T_{sw}$. Otherwise $l(sw) = l(w)-1$, by the deletion/exchange conditions for Coxeter groups, there is a reduced expression for w beginning with s. Hence $T_w = T_sT_{w'}$ for $w' = sw$ and $T_sT_w = T_s^2T_{w'} = (t^2-1)T_sT_{w'} + t^2T_{w'} = (t^2-1)T_w + t^2T_{sw}$. The final statement now follows since if $l(v) + l(w) = l(vw)$, then concatenating reduced words for v and w gives a reduced word for vw. $\square$

Note that T_w span $\mathcal{H}_W \otimes_{\mathbb{Z}} \mathbb{C}$ as a vector space.

Definition 12.2. The *Kazhdan-Lusztig involution* $i : \mathcal{H}_W \to \mathcal{H}_W$ is the algebra involution defined by $i(t) = t^{-1}$ and $i(T_w) = (T_{w^{-1}})^{-1}$.

Each T_s is invertible: $T_s^{-1} = t^{-2}T_s + (t^{-2} - 1)$ and thus the T_w are invertible as well.

Theorem 12.3 ([**?**])**.** *For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}_W$ such that*

(1) $i(C_w) = C_w$
(2) $C_w = t^{-l(w)} \sum_{x \leq w} Q_{x,w}(t) T_x$ *where $Q_{w,w} = 1$ and for $x < w$, $Q_{x,w} \in \mathbb{Z}[t]$ is a polynomial of degree $\leq l(w) - l(x) - 1$.*

More over, for each $x \leq w$ there exists $P_{x,w} \in \mathbb{Z}[q]$ such that $Q_{x,w}(t) = P_{x,w}(t^2)$, i.e. the polynomials $Q_{x,w}$ only contain terms of even degree.

This basis C_w is known as the *(dual) canonical basis* of $\mathcal{H}_W$.

Example 12.4. (1) $C_1 = 1$,
(2) $C_s = t^{-1}(T_s + 1)$ if s is a simple reflection,
(3) $C_{st} = t^{-2}(T_{st} + T_s + T_t + 1)$ if $s \neq t$ are simple reflections.

12.2. **Bruhat Decomposition.** Let G be a complex connected semisimple algebraic group. Let $\mathcal{B} = G/B$ be the flag variety and we know that $\mathcal{B} = \cup_{w \in W} BwB/B$, the Bruhat decomposition. Let $X_w^\circ = BwB/B$ then $X_w = \overline{X}_w^\circ = \cup_{v \leq w} X_v$.

This is a stratification, and since $X_w \cong \mathbb{C}^{l(w)}$, the strata are simply connected. Let IC_w be the IC-sheaf associated to the local system $\mathbb{C}$ on the orbit X_w°.

Our goal is to show the following result:

Theorem 12.5. *Let $v \leq w$ and $y \in X_v^\circ$, then $\dim_{\mathbb{C}} H^i(IC_w)_y$ is 0 if $i + l(w)$ is odd and is the coefficient of $q^{(i+l(w))/2}$ in $P_{v,w}$ if $i + l(w)$ is even.*

In particular, the coefficients are all positive.

Recall that we can also consider the G orbits on $\mathcal{B} \times \mathcal{B}$, which are all of the form $\mathfrak{X}_w^\circ = G(B/B, wB/B)$. We have $\mathfrak{X}_w^\circ = G \times^B X_w$.

By $\hat{IC}_w$ denote the IC sheaf on $\mathcal{B} \times \mathcal{B}$ that is given by the local system $\mathbb{C}$ on $\mathfrak{X}_w$. Then for $y \in \mathcal{B}$, we have

$$H^i(\hat{IC}_w)_{(B/B,y)} \cong H^{i+\dim_{\mathbb{C}}(\mathcal{B})}(IC_w)_y.$$

If $A \in D_c^b(\mathcal{B} \times \mathcal{B})$, then let

$$h(A) = \sum_{w \in W} \left(\sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}} H^i(A_w) t^i \right) T_w,$$

where A_w is the stalk of A over any point in $\mathfrak{X}_w^\circ$. I.e. $h : D_c^b(\mathcal{B} \times \mathcal{B}) \to \mathcal{H}_W$.

The above theorem will be a corollary to the following result:

Theorem 12.6.

$$C_w = t^{\dim_{\mathbb{C}} \mathcal{B}} h(\hat{IC}_w)$$

12.3. **Demazure resolutions.** The closed Schubert variety X_w is generally singular, stratified by X_u° for $u \leq w$. Of interest to us will be their Demazure resolutions. Let $s_{i_1} \cdots s_{i_n}$ be a reduced expression for a word $w \in W$. Associated to each simple reflection s_{i_j} is a minimal parabolic $P_{s_{i_j}}$. Let

$$Y_{s_i} = P_{s_{i_1}} \times^B \cdots \times^B P_{s_{i_n}}/B.$$

As an iterated $\mathbb{P}^1$ fibration it is smooth of dimension n.

There is a natural map from $Y_{s_i} \to X_w$, $m(p_1, \ldots, p_n B/B) = p_1 \ldots p_n B/B$. This map is a resolution of singularities.

As in the previous section, it will be useful to resolve $\mathfrak{X}_w$ as well. Let $\mathcal{P}_s = G/P_s$ for any simple reflection s. Then let

$$\mathfrak{Y}_{s_i} = \mathcal{B} \times_{\mathcal{P}_{s_{i_1}}} \cdots \times_{\mathcal{P}_{s_i}} \mathcal{B}.$$

Let $\pi : \mathfrak{Y}_{s_i} \to \mathfrak{X}_w$ be the proper morphism that projects on the the first and last component. This is a resolution of singularities as well and as in the last section we have $\mathfrak{Y}_{s_i} = G \times^B Y_{s_i}$.

It should be noted that due to the fibre productions in the definition of $\mathfrak{Y}_{s_i}$ it is isomorphic to the variety

$$\mathfrak{X}_{s_{i_1}} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathfrak{X}_{s_{i_n}}.$$

More over, if $l(w) \leq 2$, the resolution is an isomorphism thus X_w and $\mathfrak{X}_w$ are nonsingular in this case.

12.4. **Convolution product on $D^b_c(\mathcal{B}\times\mathcal{B})$.** Let $p_{i,j} : \mathcal{B}\times\mathcal{B}\times\mathcal{B} \to \mathcal{B}\times\mathcal{B}$ be the projection onto the i-th and j-th factor. Then given $A, B \in D^b_c(\mathcal{B} \times \mathcal{B})$, we define $A \star B \in D^b_c(\mathcal{B} \times \mathcal{B})$ to be

$$R(p_{1,3})_*(p_{1,2}^* A \overset{L}{\otimes} p_{2,3}^* B).$$

Since the projection are proper, one can check (using base change) that this product is associative and in particular the sheaf $\mathbb{C}_\Delta$ (concentrated in degree 0), is a unit. Here Δ is the diagonal in $\mathcal{B} \times \mathcal{B}$, which is also $\mathfrak{X}_1$. Note that since $\mathfrak{X}_1$ is smooth, this is just a shifted IC sheaf.

Lemma 12.7. *Let A be an object of $D^b_c(\mathcal{B} \times \mathcal{B})$ such that $H^i(A) = 0$ if i is odd (or even). Let s be a simple reflection, then $\mathbb{C}_{\mathfrak{X}_s} \star A$ has the same property. Also $h(\mathbb{C}_{\mathfrak{X}_s} \star A) = (T_s + 1)h(A)$.*

Proof.

$$
\begin{aligned}
(T_s+1)h(A) &= \sum_{w\in W}\left(\left(\sum_{i\in\mathbb{Z}}\dim_{\mathbb{C}} H^i(A_w)t^i\right)T_w + \right.\\
&\qquad \left.\left(\sum_{i\in\mathbb{Z}}\dim_{\mathbb{C}} H^i(A_w)t^i\right)T_sT_w\right)\\
&= \sum_{sw<w}\left(\sum_{i\in\mathbb{Z}}(\dim_{\mathbb{C}} H^i(A_{sw})+\dim_{\mathbb{C}} H^{i-2}(A_w))t^i\right)T_w+\\
&\qquad \sum_{sw>w}\left(\sum_{i\in\mathbb{Z}}(\dim_{\mathbb{C}} H^i(A_w)+\dim_{\mathbb{C}} H^{i-2}(A_{sw}))t^i\right)T_w
\end{aligned}
$$

Here we apply the formulas from Lemma 12.1 and then collect terms by T_w.

To show the final statement in the theorem, it thus suffices to show that $\dim_{\mathbb{C}} H^i((\mathbb{C}_{\mathfrak{X}_s}\star A)_w)$ is $\dim_{\mathbb{C}} H^i(A_{sw})+\dim_{\mathbb{C}} H^{i-2}(A_w)$ when $sw<w$ and $\dim_{\mathbb{C}} H^i(A_w)+\dim_{\mathbb{C}} H^{i-2}(A_{sw})$ when $sw>w$.

Consider the fibre square for the inclusion of $(B/B, wB/B)$ into $\mathcal{B}\times\mathcal{B}$ and the map $p_{1,3}$. The fibre product is $Z_w = \{(B/B, b, wB/B)|b\in\mathcal{B}\}$. By base change the stalk is thus the global sections of the restriction of $p_{1,2}^*\mathbb{C}_{\mathfrak{X}_s}\overset{L}{\otimes} p_{2,3}^*A$ to Z_w. But since $p_{1,2}^*\mathbb{C}_{\mathfrak{X}_s}$ is supported on $\mathfrak{X}_s\times\mathcal{B}$, the global sections of $(p_{1,2}^*\mathbb{C}_{\mathfrak{X}_s}\overset{L}{\otimes} p_{2,3}^*A)|_{Z_w}$ only have support over $Z_w\cap\mathfrak{X}_s\times\mathcal{B}$, i.e. over the set $Z_w^s=\{(B/B, gB/B, wB/B)|g\in P_s\}$

Note that Z_w^s is isomorphic to $\mathbb{P}^1$ and by the above, if C is the restriction of the above tensor product to Z_w^s, then $H^i((\mathbb{C}_{\mathfrak{X}_s}\star A)_w)\cong H^i(Z_w^s, C)$.

We will want to know the stalk of C above each point on Z_w^s. The stalk at a point $(B/B, gB/B, wB/B)$ for $g\in P_s$ can be computed by

$$(\mathbb{C}_{\mathfrak{X}_s})_{(B/B,gB/B)}\otimes A_{(gB/B,wB/B)}.$$

Since $(B/B, gB/B)\in\mathfrak{X}_s$, the left side is just the complex $\mathbb{C}$ concentrated in degree 0. The right side depends on which orbit $(gB/B, wB/B)$ lies in. If $sw<w$, then $(gB/B, wB/B)\in\mathfrak{X}_{sw}^\circ$ iff $gB=sB$. The remaining points all lie in $\mathfrak{X}_w^\circ$. If $sw>w$, the behaviour is the opposite.

If ι is the inclusion of this point into Z_s^w and j the inclusion of the complement, then we consider the triangle:

$$j_!j^*C\to C\to R\iota_*\iota^*C\to .$$

This gives rise to a long exact sequence:

$$
\begin{aligned}
\cdots\to H^{i-1}(Z_s^w, R\iota_*\iota^*C)\to H^i(Z_s^w, j_!j^*C)\to H^i(Z_s^w, C)\to&\\
H^i(Z_s^w, R\iota_*\iota^*C)\to H^{i+1}(Z_s^w, j_!j^*C)&\to\cdots
\end{aligned}
$$

By the above calculations, ι^*C is simply the stalk of A at sw, so $H^i(Z_w^s, R\iota_*\iota^*C)\cong H^i(\{pt\}, A_{sw}) = H^i(A_{sw})$. Note that when $H^i(A)=0$ we have $H^i(A_{sw})=0$.

Consider taking the stalk of j^*C at any point on $Z^*_w - \{(B/B, sB/B, wB/B)\} \cong \mathbb{C}$. Again, by the above computations, the fibre is the complex A_w. Thus using the fact that $\mathbb{C} \cong \mathbb{C} \times \{pt\}$, we have $j^*C \cong \mathbb{C}_{\mathbb{C}} \overset{L}{\boxtimes} A_w$. Since Z^s_w is compact,

$$H^i(Z^s_w, j_! j^* C) \cong H^i_c(Z^s_w, j_! j^* C) \cong H^i_c(\mathbb{C}, j^*C) \cong H^i_c(\mathbb{C} \times \{pt\}, \mathbb{C}_{\mathbb{C}} \overset{L}{\boxtimes} A_w).$$

We can apply the K unneth formula to the right most term, and since $H^i_c(\mathbb{C}, \mathbb{C})$ is $\mathbb{C}$ when $i = 2$ and 0 otherwise, we have $H^i(Z^s_w, j_! j^* C) \cong H^{i-2}(A_w)$.

The above long exact sequence now falls apart into short exact sequences of the form

$$0 \to H^i(Z^s_w, C) \to 0$$

or

$$0 \to H^{i-2}(A_w) \to H^i(Z^s_w, C) \to H^i(A_{sw}) \to 0,$$

depending on the parity of i.

The results of the lemma now follow. □

12.5. **Proof of theorem 12.6.** We can now prove the desired theorem.

Proof. We induct bases on the length of w.

If $l(w) = 0$, then $\hat{IC}_w \cong \mathbb{C}_{\mathfrak{X}_1}[\dim_{\mathbb{C}} \mathcal{B}]$. Thus $h(\hat{IC}_w) = t^{-\dim_{\mathbb{C}} \mathcal{B}} T_1 = t^{-\dim_{\mathbb{C}} \mathcal{B}} C_1$.

If $l(w) = 1$, then $w = s$ a simple reflection and $\hat{IC}_s \cong \mathbb{C}_{\mathfrak{X}_s}[1 + \dim_{\mathbb{C}} \mathcal{B}]$. Thus $h(\hat{IC}_s) = t^{-1-\dim_{\mathbb{C}} \mathcal{B}} T_s + t^{-1-\dim_{\mathbb{C}} \mathcal{B}} T_1 = t^{-1-\dim_{\mathbb{C}} \mathcal{B}}(T_s + 1) = t^{-\dim_{\mathbb{C}} \mathcal{B}} C_s$.

Let $s_1 \cdots s_n$ be a reduced expression for w, the consider the Demazure resolution $\pi : \mathfrak{Y}_{s_1 \ldots s_n} \to \mathfrak{X}_w$.

Now $IC(\mathfrak{Y}_{s_1 \ldots s_n}) = \mathbb{C}_{\mathfrak{Y}_{s_1 \ldots s_n}}[n + \dim_{\mathbb{C}} \mathcal{B}]$. The map π is not in general semi-small, but we can apply the weakest form of the decomposition theorem to it.

$$R\pi_* C_{\mathfrak{Y}_{s_1 \ldots s_n}}[n + \dim_{\mathbb{C}} \mathcal{B}] \cong \oplus_{u \leq w} \hat{IC}_u \times V_u$$

where V_u is a graded $\mathbb{C}$ vector space where the grading handles the shifts of the $\hat{IC}_u$ which appear.

The sheaf $\hat{IC}_w$ can only appear once and it does so with no shift, thus $V_w = \mathbb{C}$ with a 0 grading. This follows because π is an isomorphism over the open stratum, hence restricting the pushforward to this stratum will result in a complex whose cohomology only exists in degree $-n - \dim_{\mathbb{C}} \mathcal{B}$.

Otherwise, since the sheaf we pushforward is Verdier self dual and π is proper, the resulting complex is self dual as well. This implies that the j-th graded piece of V_u is equal to the $-j$-th graded piece of V_u. These two facts together along with the fact that $h(A \oplus B) = h(A) + h(B)$, gives

$$h(R\pi_* C_{\mathfrak{Y}_{s_1 \ldots s_n}}[n + \dim_{\mathbb{C}} \mathcal{B}]) = h(\hat{IC}_w) + \sum_{u<w} Q_u(t) h(\hat{IC}_u).$$

Here $Q_u(t)$ is simply the grading polynomial of V_u and hence satisfies $Q_u(t) = Q_u(t^{-1})$.

By induction $h(\hat{IC}_u) = t^{-\dim_{\mathbb{C}} \mathcal{B}} C_u$, thus

$$t^{\dim_{\mathbb{C}} \mathcal{B}} \sum_{u<w} Q_u(t) h(\hat{IC}_u)$$

is fixed by the involution on $\mathcal{H}_W$.

Since

$$\mathfrak{Y}_{s_1 \ldots s_n} \cong \mathfrak{X}_{s_1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathfrak{X}_{s_n},$$

it follows that

$$R\pi_* \mathbb{C}_{\mathfrak{Y}_{s_1 \ldots s_n}} \cong \mathbb{C}_{\mathfrak{X}_{s_1}} \star \cdots \star \mathbb{C}_{\mathfrak{X}_{s_n}}.$$

Thus by previous lemma, we have

$$h(R\pi_* \mathfrak{Y}_{s_1 \cdots s_n}[n+\dim_{\mathbb{C}} \mathcal{B}]) = t^{-n-\dim_{\mathbb{C}} \mathcal{B}}(1+T_{s_1}) \cdots (1+T_{s_n}) = t^{-\dim_{\mathbb{C}} \mathcal{B}} C_{s_1} \cdots C_{s_n}.$$

Hence $t^{\dim_{\mathbb{C}} \mathcal{B}}(R\pi_* \mathfrak{Y}_{s_1 \cdots s_n}[n + \dim_{\mathbb{C}} \mathcal{B}])$ is also fixed by the involution.

This implies that $t^{\dim_{\mathbb{C}} \mathcal{B}} h(\hat{IC}_w)$ is fixed by the involution as well.

The remaining condition is on the degrees of the polynomial which occurs as the coefficient of T_u in $t^{\dim_{\mathbb{C}} \mathcal{B}} h(\hat{IC}_w)$, but this follows from the general properties of IC complexes. $\square$

12.6. **examples.**

12.6.1. A_1. $G = SL_2(\mathbb{C})$, $W = S_2$. We know the canonical basis elements in $\mathcal{H}_W$ are $C_1 = 1$ and $C_{s_1} = t^{-1}(T_{s_1} + 1)$. The flag variety G/B is simply a copy of $\mathbb{CP}^1$. The two Schubert cells are $X_1^\circ = \{pt\}$ and $X_{s_1}^\circ = \mathbb{C}$. The Schubert varieties are both smooth, $X_1 = \{pt\}$ and $X_{s_1} = \mathbb{CP}^1$. Hence the $IC_1 = \mathbb{C}_{X_1}[0]$ and $IC_{s_1} = \mathbb{C}_{X_{s_1}}[1]$.

12.6.2. A_2. $G = SL_3(\mathbb{C})$ and $W = S_3$. The only C_w that we do not already know is $C_{s_1 s_2 s_1}$. Now, if $s_1 \cdots s_n$ is a reduced word for w, then

$$C_{s_1} C_{s_2} \cdots C_{s_n} = C_w + \text{ lower order terms.}$$

This is clear, since the term T_w appears with coefficient t^{-n} in both C_w and the left hand side.

$$C_{s_1} C_{s_2} C_{s_1} = t^{-3}(T_{s_1}+1)(T_{s_2}+1)(T_{s_1}+1) = t^{-3}(T_{s_1 s_2 s_1}+T_{s_1 s_2}+T_{s_2 s_1}+(t^2+1)T_{s_1}+T_{s_2}+(t^2+1)))$$

We note that the right hand side cannot be C_w, since the coefficient of T_{s_1} violates the degree bound of $l(s_1 s_2 s_1) - l(s_1) - 1 = 1$. Thus we must subtract off C_{s_1}, this gives

$$C_{s_1} C_{s_2} C_{s_1} - C_{s_1} = t^{-3}(T_{s_1 s_2 s_1} + T_{s_1 s_2} + T_{s_2 s_1} + T_{s_1} + T_{s_2} + 1) = C_{s_1 s_2 s_1}.$$

One can check that if $l(w) \leq 2$, the Demazure resolution is an isomorphism, so that $X_{s_1 s_2}$ and $X_{s_2 s_1}$ are both smooth. In fact they are just $\mathbb{CP}^1$ bundles over $\mathbb{CP}^1$. The Schubert variety $X_{s_1 s_2 s_1} \cong G/B$ so it is also smooth. Thus the IC sheaves in each case are just $\mathbb{C}[l(w)]$. This agrees with the calculation of C_w above.

The Demazure resolution is not an isomorphism for $w = s_1 s_2 s_1$, so what is $Rm_* IC(Y_{s_1 s_2 s_1})$? This should be the same as $R\pi_* IC(\mathfrak{Y}_{s_1 s_2 s_1})$. Since

$h(R\pi_* IC(\mathfrak{Y}_{s_1s_2s_1})) = t^{-\dim_{\mathbb{C}}\mathcal{B}} C_{s_1}C_{s_2}C_{s_1}$, the same method that allows us to compute that $C_{s_1}C_{s_2}C_{s_1} = C_{s_1} + C_{s_1s_2s_1}$ will show that $Rm_* IC(Y_{s_1s_2s_1}) = IC_{s_1} + IC_{s_1s_2s_1}$.

From a geometric perspective, $\pi : \mathfrak{Y}_{s_1s_2s_1} \to \mathfrak{X}_{s_1s_2s_1}$ is an isomorphism over $\mathfrak{X}_{s_1s_2s_1} - \mathfrak{X}_{s_1}$ and has a $\mathbb{CP}_1$ fibre over the remaining points. In this case π is semismall.

12.6.3. B_2. Note that $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2 = s_2s_1s_2s_1\}$. If we consider $s_1s_2s_1$, then as above, $C_{s_1s_2s_1} = t^{-3}(T_{s_1s_2s_1} + T_{s_1s_2} + T_{s_2s_1} + T_{s_1} + T_{s_2} + 1))$. Hence $IC_{s_1s_2s_1} = \mathbb{C}_{X_{s_1s_2s_2}}[3]$. But one can check that $X_{s_1s_2s_2}$ is singular with singular locus X_{s_1}. Thus we see that it is possible to have an IC sheaf which does not detect singularities.

12.6.4. A_3. Here we will consider the word $w = s_1s_3s_2s_3s_1$. If one computes $C_{s_1}C_{s_3}C_{s_2}C_{s_3}C_{s_1}$, for most $u \leq w$, the coefficient of T_u is 1. The exception is if $u \in \{s_1s_3, s_1, s_3, 1\}$ in which case it is $t^4 + 2t^2 + 1$. Thus we know the cohomology of the stalks of the sheaf $Y_{s_1s_3s_2s_3s_1}$. Note that in order for $Rm_*\mathbb{C}_{Y_{s_1s_3s_2s_3s_1}}[5]$ to be a perverse sheaf, the cohomology of the stalks over $X^\circ_{s_1s_3}$ must vanish in degree > -2. But the fact that we have a term t^4 tells us that it does not.

In particular, $Rm_*\mathbb{C}_{Y_{s_1s_3s_2s_3s_1}}[5] = A \oplus IC_{s_1s_3}[-1]$, But since the left hand side is Verdier self dual, there must also be a factor of $IC_{s_1s_3}[1]$ in A. This leaves us with $A = IC_{s_1s_3}[1] \oplus B$ where B is now Verdier self dual. Note that it also satisfies the support conditions of an IC sheaf and thus it is an IC sheaf it self. Since the cohomology of its stalks is $\mathbb{C}$ over $X^\circ_{s_1s_3s_2s_3s_1}$ in degree -5, it follows that this is the $IC_{s_1s_3s_2s_3s_1}$.

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