MATH 2310 — Final

December 6, 2012

Name: ____

- Do not open this booklet until instructed to begin.
- You will have a total of 150 minutes to complete the exam, which consists of 10 problems. Please show work and/or justification if asked (although doing so even when not asked may accumulate some partial credit). Each problem is weighted equally. Books, notes, calculators, cell phones, and other forms of assistance are not to be used during the exam.
- Each problem appears on its own page. Please feel free to use the back of the page to continue your work; if you require additional paper, raise your hand and I will supply some. Label clearly which problem appears on the additional page(s) and indicate the final answer.

Problem	Grade	Possible		Problem	Grade	Possible
1		10		6		10
2		10		7		10
3		10		8		10
4		10		9		10
5		10		10		10
				Total		100

Problem 1:
Consider for each real number r the matrix

$$A_r = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & r \end{bmatrix}.$$

(a) Compute $\det(A_r)$.

(b) Find all real numbers r such that A_r is invertible.

(c) Compute $(A_2)^{-1}$, if possible (where A_2 is just A_r with $r = 2$).

Problem 2:

Consider the subspace $V \subseteq \mathbb{R}^4$ spanned by the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

(a) Find an orthonormal basis for V.

(b) Find a basis for the orthogonal complement $V^{\perp}$.

(c) Find the vector in V closest to the vector $\begin{bmatrix} -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}$.

Problem 3:

True/False section! No written justification is required; just circle T or F (but not both!). Each correct answer is worth one point, each blank answer is worth one half point, and each incorrect answer is worth no points.

T F A square matrix is invertible if and only if it does not have 0 as an eigenvalue.

T F The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$.

T F If A is a square matrix, then $\det(3A) = 3\det(A)$.

T F If A and B are matrices for which AB is defined, then we always have $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

T F If A is an $(m \times n)$ matrix with rank m, then $A\mathbf{x} = \mathbf{b}$ has at least one solution for every possible $\mathbf{b}$.

T F The set of vectors (x_1, x_2, x_3) with $x_1 + x_2 - x_3 = 1$ forms a subspace of $\mathbb{R}^3$.

T F A square, upper triangular matrix with (strictly) negative entries along the diagonal is always invertible.

T F The only vector in $\mathbb{R}^4$ whose dot product with every vector is 0 is the zero vector.

T F The transpose of an invertible matrix is again invertible.

T F All Markov matrices with nonzero entries are symmetric.

Problem 4:
Consider the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

(a) Compute B^{33}. The answer should be a single matrix, but it's ok to leave real number exponents in it.

(b) Find a matrix $C \neq \pm B$ (that is, C can be neither $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ nor $\begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix}$) with real entries such that $C^2 = B^2$.

Problem 5:

Consider the following assortment of vectors and matrices:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Evaluate the following, if the operation is defined.

(a) $\mathbf{v}_1 \cdot \mathbf{v}_2$

(b) $\mathbf{v}_1 \mathbf{v}_2^{\mathrm{T}}$

(c) $\mathbf{v}_2 \cdot \mathbf{v}_2$

(d) $M_1 M_2$

(e) $M_2 M_1$

(f) $2\mathbf{v}_1 - M_1 \mathbf{v}_1$

Problem 6:

Find matrices U, V, and Σ such that $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} = U\Sigma V^{\mathrm{T}}$, both U and V are matrices with orthonormal columns, and Σ is a diagonal matrix of singular values.

Problem 7:

We now work with the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

(a) Put M into reduced row echelon form.

(b) What is the rank of M?

(c) Find a vector $\mathbf{b} \in \mathbb{R}^3$ such that the system $M\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} \in \mathbb{R}^3$, or explain why no such vector exists.

Problem 8:

A frog is hopping through a huge field. Look out! There are wells everywhere! When the frog is not in a well, he has a 10% chance of falling into one within the next minute. When the frog is in a well, there is a 50% chance he escapes within the next minute.

(a) Write a Markov matrix modeling this situation.

(b) Find the eigenvalues of the matrix in the previous part.

(c) Is there a unique steady state vector for this system? If so, compute the long-run probability that the frog is in a well at any given moment.

Problem 9:

Find an equation for the line which best fits (in the sense of least squares approximation) the data set $\{(0,0),(1,2),(2,4),(3,1)\}$. These data points are given in the format (x,y) (or, if you prefer the book's notation, (t,b)).

Problem 10:
Consider the matrices

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find a single (nonzero) vector $\mathbf{v} \in \mathbb{R}^3$ which is an eigenvector for both J and K (possibly paired with different eigenvalues for each matrix).