Math 2310 (001 and 002)
Final Exam (December 13, 2013)

NAME: ________________________

- Time: you have 150 minutes (9:00AM – 11:30AM).
- Show all your work and justifications.
- The use of calculators or notes is not permitted during the exam.
- You may use both sides of the paper. There are extra blank pages at the end if you need more space.
- Make sure you have **11 pages and 8 problems** before starting the exam.

Academic integrity is expected of all students of Cornell University at all times.
Understanding this, I declare I shall not give, use, or receive unauthorized aid.

SIGNATURE: ________________________

Problem 1: ____ / 25
Problem 2: ____ / 25
Problem 3: ____ / 25
Problem 4: ____ / 25
Problem 5: ____ / 25
Problem 6: ____ / 25
Problem 7: ____ / 25
Problem 8: ____ / 25

Total: ____ / 200

Problem 1: Decide whether each statement below is **True** or **False**?
(no justification is needed; just **True** or **False** in front of each statement)

(1) For any two *unit* vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we must have $|\mathbf{u} \cdot \mathbf{v}| \leq 1$.

(2) If the sizes of matrices A and B are so that AB is well-defined, then $(AB)^T$ is always the same as $B^T A^T$.

(3) All the eigenvalues of a 21×21 real matrix can be complex non-real numbers (i.e. numbers of the form $a + bi$ with $b \neq 0$).

(4) If A is a 2×2 matrix, then $\det(2A) = 2\det(A)$.

(5) If the sizes of matrices A and B are so that AB is well-defined, then

$$\mathrm{rank(AB)} \leq \mathrm{min(rank(A), rank(B))} \ .$$

(6) If $A = (a_{ij})$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots \lambda_n$ then $\mathrm{Trace}(A) = \sum_{i=1}^{n} a_{ii}$ is equal to

$$\sum_{i=1}^{n} \lambda_i \ ,$$

and $\det(A)$ is equal to

$$\prod_{i=1}^{n} \lambda_i \ .$$

(7) Let A and B be $n \times n$ matrices. If α is an eigenvalue of A and β is an eigenvalue of B then $\alpha + \beta$ is an eigenvalue of $A + B$ and $\alpha\beta$ is an eigenvalue of AB.

(8) The set of all solutions to the equation $x + 2y + 3z = 4$ forms a *subspace* of $\mathbb{R}^3$.

(9) A *lower triangular* matrix with no zeros on its main diagonal is invertible.

(10) According to the *Cauchy-Binet formula* we must have

$$\det(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}) = (\det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix})^2 + (\det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix})^2 + (\det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix})^2 \ .$$

Problem 2: Consider the system of equations

$$\begin{cases} x - 2y + az = 2 \\ x + y + z = 0 \\ 3y + z = 2 \end{cases}$$

(a) For which values of a, if any, does this system have a unique solution?

(b) For which values of a, if any, does this system have no solution?

(c) For which values of a, if any, does this sytem have infinitely many solutions?

Problem 3: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix} \ .$$

Apply *Gram–Schmidt* algorithm to the columns of A and write

$$A = QR \ ,$$

where Q is an *orthogonal matrix* (i.e. $Q^T Q = I$) and R is an *upper triangular* matrix. (*Hint*: columns of Q are just the output vectors of the Gram–Schmidt algorithm, and $R = Q^T A$).

Problem 4: We think the relation between the input and output of a system is a degree one polynomial. In other words, we expect on input x the output should be of the form $y = ax + b$. But we do not know the real numbers a and b. We collected some data $(\text{input}, \text{output}) = (x, y)$ and obtained:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, 5) \ .$$

(a) Is there a line $y = ax + b$ that matches this data? If not, what is the "best" solution for a and b?

(Recall "best" solution means the *least squares solution*, i.e. the solution that minimizes the norm of the error vector.)

(b) What is the norm of the *error vector*?

Problem 5: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & -3 \end{pmatrix} .$$

(a) Find the *eigenvalues* of A.

(b) Find a *basis* for each *eigenspace*.

(c) What is the *algebraic multiplicity* and *geometric multiplicity* of each eigenvalue?

(d) Is A *diagonalizable*? If it is, write down an *invertible* matrix S and a *diagonal* matrix Λ so that $A = S\Lambda S^{-1}$.

(e) Compute A^{2014}. (It is ok to have numbers of the form a^m in your final answer!)

Problem 6: Consider the *symmetric* matrix

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

(a) Compute the eigenvalues and an *orthonormal* basis for each eigenspace.

(b) Write down an *orthogonal* matrix Q and a *diagonal* matrix Λ so that $B = Q\Lambda Q^T$.

(d) Write down a matrix R so that $B = RR^T$.

(e) Let P_λ denote the *projection* matrix onto the eigenspace of λ. For *distinct* eigenvalues λ, compute P_λ and check

$$B = \sum \lambda P_\lambda .$$

(Recall if the columns of a matrix Q are orthonormal, the projection matrix onto $\mathrm{Col}(Q)$ can be computed by $P = QQ^T$.)

Problem 7: Let R denote the $m \times m$ *reflection matrix* with respect to a vector $\mathbf{u}$ in $V = \mathbb{R}^m$.

(a) What are the *eigenvalues* and *eigenspaces* of the matrix R?

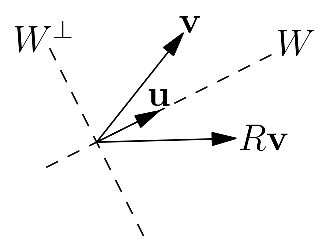
(b) What is the *geometric multiplicity* of each eigenvalue?

(c) Is R *diagonalizable*? What is the *algebraic multiplicity* of each eigenvalue?

(d) What is the *trace* of R?

(e) What is the *determinant* of R?

(*Hint*: Although you might remember a formula for computing reflection matrices, the key to this problem is to **not** use that formula, and instead use the geometric intuition to solve (a).



Parts (b)–(e) can all be answered once you know (a).)

Problem 8: Assume A is a *positive definite* $m \times m$ matrix, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$ in $\mathbb{R}^m$. Let B be any $m \times n$ matrix and consider the $n \times n$ matrix

$$C = B^T AB \ .$$

(a) Show that $\mathbf{y}^T C\mathbf{y} \geq 0$ for all vectors $\mathbf{y}$ in $\mathbb{R}^n$ (such a matrix C is called *positive semidefinite*).

(a) If we assume $\text{rank}(B) = n$ (or, equivalently, $\text{Null(B)} = \{\mathbf{0}\}$), show that C is indeed *positive definite*, i.e. $\mathbf{y}^T C\mathbf{y} > 0$ for all *nonzero* vectors $\mathbf{y}$ in $\mathbb{R}^n$.

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