Math 2310 Final exam, May 13, 2014 **Name:**

Work + Communication = Credit

The two promised formulas:

$$P\,\vec{b} = A(A^TA)^{-1}A^T\,\vec{b}.$$

$$C = c - \frac{B^Tc}{B^TB}B - \frac{A^Tc}{A^TA}A.$$

1. (15 points) For the system of equations below determine the all of the value(s) of b such that the system is consistent.

$$\begin{array}{ccccccccc} x & + & 2y & - & z & & & = & 4 \\ 3x & & & & 2z & + & w & = & -1 \\ x & - & 4y & + & 4z & + & w & = & b \end{array}.$$

2. (16 points) Are the following vectors independent? If so, show work that demonstrates the vectors are independent. If not, write one vector as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

3. (16 points) Below are four subsets of known vector spaces. Two of them are subspaces, two are not. Identify those that are not subspaces and write one or two sentences explaining why they are not subspaces.

(a) All vectors $\vec{x}$ in $\mathbb{R}^3$ such that $\vec{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$ and $\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$.

(b) All vectors $\vec{x}$ in $\mathbb{R}^3$ such that $\vec{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$ or $\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$.

(c) The set of all 2×2 matrices A such that $A = A^T$.

(d) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbb{R}^4$ such that $4x_1 - 8x_2 + 23x_3 - \sqrt{\pi}x_4 = 17$.

4. (15 points) Let A be the Markov matrix

$$\begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}.$$

Let $\vec{u}_0 = \begin{bmatrix} x \\ y \end{bmatrix}$ be an initial vector in $\mathbb{R}^2$ with $x + y = 150$. What is $\lim_{k\to\infty} A^k \vec{u}_0$?

5. (16 points) Let V be the subspace of $\mathbb{R}^5$ consisting of all vectors orthogonal to $\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}$.

(a) Write down a matrix A such that V is the null space of A.

(b) Find a basis for V.

(c) Write down a matrix B so that V is the row space of B.

6. (14 points)

(a) Let P be the projection matrix for a 4-dimensional subspace of $\mathbb{R}^6$. P can be diagonalized as $P = S\Lambda S^{-1}$. This is already enough information to determine Λ if we insist that $\lambda_1 \geq \cdots \geq \lambda_6$.

$$\Lambda = ?$$

(b) Let V be a 5 dimensional subspace of $\mathbb{R}^7$. Recall that an 7×7 matrix A is a reflection matrix for V if A is a matrix such that $A\vec{v} = v$ for all vectors $\vec{v}$ in V and $A\vec{w} = -\vec{w}$ for all vectors $\vec{w}$ in $V^{\perp}$. Any reflection matrix can be diagonalized so write $A = S\Lambda S^{-1}$. This is already enough information to determine Λ if we insist that $\lambda_1 \geq \cdots \geq \lambda_7$.

$$\Lambda = ?.$$

7. (16 points) For this problem A is the 3×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & -2 \\ 2 & 1 & 0 & -3 \end{bmatrix}.$$

(a) Find a basis for the null space of A.

(b) Find an orthonormal basis for the null space of A.

(c) Find an orthonormal basis for the column space of A.

8. (16 points) Let A be the 2×2 matrix

$$\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}.$$

Compute the first term of the singular value decomposition

$$A = \vec{u}_1 \sigma_1 \vec{v}_1 + \vec{u}_2 \sigma_2 \vec{v}_2, \ \ \sigma_1 \geq \sigma_2.$$

9. (16 points)

(a) Compute the determinant of

$$\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 2 & -\pi \\ -2 & \sqrt{43} & e^2 & 53 \end{bmatrix}.$$

(b) Is the determinant of an orthogonal matrix always one or minus one? If so, use at most three sentences to explain why. If not, write down an orthogonal matrix which has a determinant other than one or minus one.

(c) True of False (no explanation required): If Q is an orthogonal matrix, then $Q^{-1} = Q^T$.

(d) Let $\vec{v}_1, \ldots, \vec{v}_m$ be an orthonormal basis of a subspace V of $\mathbb{R}^n$ and suppose A is the matrix whose columns are these vectors.

$$A = \left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \end{array} \right].$$

True or False: The projection matrix for V is AA^T. If true give a short explanation. If false, give an example.