MATH 2310 — Prelim 1

October 3, 2012

Name: ____

- Do not open this booklet until instructed to begin.
- You will have a total of 50 minutes to complete the exam, which consists of 5 problems. Please show work and/or justification when asked (although doing so even when not asked may accumulate some partial credit). Each problem is weighted equally. Books, notes, calculators, cell phones, and other forms of assistance are not to be used during the exam.
- Each problem appears on its own page. Please feel free to use the back of the page to continue your work; if you require additional paper, raise your hand and I will supply some. Label clearly which problem appears on the additional page(s) and indicate the final answer.

Problem	Grade	Possible
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Find A^{-1} if it exists.

(b) Find all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Problem 2:

Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 7 & 8 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 3 & 4 & 3 & 4 & 5 & 2 \end{bmatrix}.$$

(a) Exactly one of these matrices is the reduced row echelon form of B. Circle the correct one.

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 6 & 2 & 4 \\ 5 & 7 & 2 & 5 \\ 6 & 8 & 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 & -1 & -6 \\ 0 & 1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -2 & -6 & -8 & -10 & -16 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) What is the rank r of B?

(c) Write the column space $C(B)$ as a span of a set of r vectors.

(d) Write the null space $N(B)$ as a span of a set of $6 - r$ vectors.

Problem 3:

True/False section! No written justification is required; just circle T or F (but not both!). Each correct answer is worth two points, each blank answer is worth one point, and each incorrect answer is worth no points.

T F If A is a symmetric matrix, then $A^2 = AA$ is symmetric.

T F If A and B are matrices for which AB is defined, then $\text{rank}(AB) = \text{rank}(B)$.

T F If A is an $(m \times n)$ matrix with rank m, then $A\mathbf{x} = \mathbf{b}$ has at least one solution for every possible $\mathbf{b}$.

T F The set of vectors (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 1$ forms a subspace of $\mathbb{R}^3$.

T F A square, upper triangular matrix with (strictly) positive entries along the diagonal is always invertible.

T F The only vector in $\mathbb{R}^2$ whose dot product with every vector is 0 is the zero vector.

T F The transpose of an invertible matrix is again invertible.

T F The adjacency matrix of any (undirected) graph (with no self-loops or multiple edges, i.e., those we talked about in class) is a symmetric matrix containing only 0s and 1s with every diagonal entry equal to 0.

T F All powers (i.e., matrices of the form $A^n = \underbrace{AA \cdots A}_{n \text{ times}}$) of an adjacency matrix of a graph are again adjacency matrices of graphs.

T F It is impossible for an adjacency matrix of a graph to have full rank.

Problem 4:

Let I denote the (3×3) identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let J denote the (3×3) matrix with every entry equal to 1, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Compute I^2, IJ, JI, and J^2.

(b) Suppose that a and b are real numbers. Compute the product

$$(I + aJ)(I + bJ).$$

(c) For what values of a is $I + aJ$ invertible? For such values, what is the inverse of $I + aJ$? (*Application* Quickly compute matrix inverses in class to impress students.)

Problem 5:
Define the matrices A, B and vectors $\mathbf{x}, \mathbf{y}$ by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Compute the following products, if possible.

(a) AB.

(b) $A(B^{\mathrm{T}})$.

(c) $B\mathbf{x}$.

(d) $(\mathbf{x}^{\mathrm{T}})\mathbf{y}$.

(e) $\mathbf{x}(\mathbf{y}^{\mathrm{T}})$.