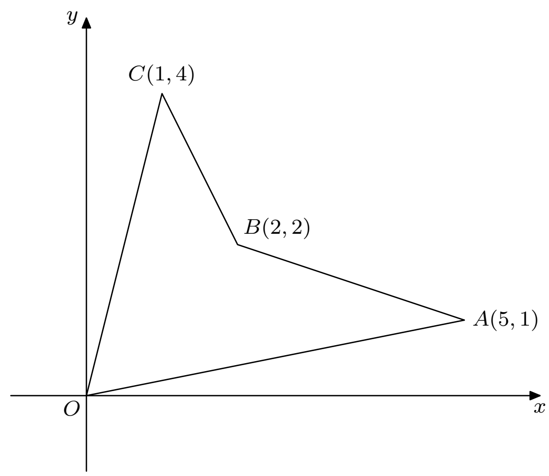
Math 2310 Take-home prelim 2

Due Monday 16 November

You should hand in your solutions in class on Monday 16 November. This prelim will count towards your final grade. There are 8 questions in total. You are supposed to work on the problems on your own.

1. Find the area of the quadrilateral $OABC$ on the figure below, coordinates given in brackets. [See pp. 160—163 of the book.]



2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 1 \end{bmatrix}$$

   (a) Calculate the nullspace of the matrix A.

   (b) Let $B = A^T$. Find the rank of B.

   (c) Find a basis for the column space of B.

3. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

   (a) Find the reduced row echelon form of A.

   (b) Do the rows of A span $\mathbb{R}^3$? Explain your answer.

   (c) Do the columns of A span $\mathbb{R}^3$? Explain your answer.

   (d) Your friend Bob claims that there exist bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbb{R}^3$ such that $[\mathbf{x}]_S = A[\mathbf{x}]_T$ for all $\mathbf{x}$ in $\mathbb{R}^3$. Explain why this cannot possibly be true.

4. Let A be an $n \times n$ matrix with integer entries.

   (a) If $\det(A) = 1$, show that A^{-1} has integer entries.

(b) Suppose A^{-1} has integer entries. What are the possibilities for $\det(A)$? Explain.

5. Find out whether the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

form a basis in the space of all 2×2 matrices.

6. Find all vectors in $\mathbb{R}^3$ of length ≤ 2 with integer entries. Which of them are orthogonal to the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$?

7. The population of sapsuckers in Sapsucker Woods is described by the following model. Let c_k denote the number of chicks in year k, let j_k denote the number of juveniles in year k, and let a_k denote the number of adults in year k. Then

$$\begin{bmatrix} c_{k+1} \\ j_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2 \\ 0.25 & 0.875 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} c_k \\ j_k \\ a_k \end{bmatrix}$$

Let A be the matrix

$$A = \begin{bmatrix} 0 & 0 & 0.2 \\ 0.25 & 0.875 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix}$$

(a) A vector $\mathbf{v}$ in $\mathbb{R}^3$ is called a *steady-state vector* of A if $A\mathbf{v} = \mathbf{v}$. Explain what this means in terms of the model.

(b) Find all steady-state vectors for A.

(c) After heavy logging in Sapsucker woods, biologists find that the model is no longer accurate. Instead, a more suitable model is

$$\begin{bmatrix} c_{k+1} \\ j_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} c_k \\ j_k \\ a_k \end{bmatrix}$$

Under this new model, what do you think will happen to the population of sapsuckers in the long term? Explain your answer.

8. Let

$$A = \begin{bmatrix} 3 & 5 & 7 & 3 & 2 \\ 2 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 5 & 2 \end{bmatrix}$$

(a) Calculate $\det(A)$.

(b) Is A invertible? Explain your answer.

(c) Calculate $\det(AA^T)$.