Math 2310 (Lecture 001)
Midterm Exam 2 (November 15, 2013)

NAME: ________________________

- Time: you have 50 minutes (12:20-1:10PM).
- Show all your work and justifications.
- The use of calculators or notes is not permitted during the exam.
- You may use both sides of the paper.

Academic integrity is expected of all students of Cornell University at all times. In particular, **you are not allowed to discuss the exam with the students of the upcoming class (1:25-2:15 session) after you leave this room.**

Understanding this, I declare I shall not give, use, or receive unauthorized aid.

SIGNATURE: ________________________

Problem 1: ____ / 25
Problem 2: ____ / 25
Problem 3: ____ / 25
Problem 4: ____ / 25

Total: ____ / 100

Problem 1: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & 1 & 4 & 3 \end{pmatrix} .$$

(a) Find a *basis* for each of the following subspaces:

- Col(A) (i.e. the *column space* of A).
- Row(A) (i.e. the *row space* of A).
- Null(A) (i.e the *null space* of A).

(b) What is the *rank* of A?

(c) What is the *orthogonal complement* of Null(A) in $\mathbb{R}^4$?

Problem 2: Consider the matrix

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} .$$

(a) Show that the columns of Q form an orthonormal basis for the subspace $W = \mathrm{Col}(Q)$ of $\mathbb{R}^4$.

(b) Compute the *projection* of

$$\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

onto W.

Problem 3: Let A be an $m \times n$ matrix. Recall **if** the columns of A are independent, the projection matrix onto $W = \mathrm{Col}(A)$ can be computed by

$$P = A(A^T A)^{-1} A^T \ .$$

Note that P is a square $(m \times m)$ matrix.

(a) If the columns of A are **not** independent, how would you compute P?

(b) Show that for any projection matrix P, we have $P^2 = P$.

(c) Let $\det(P) = x$. The following argument suggests that x is always 1, which is incorrect.

$$P = A(A^T A)^{-1} A^T \quad \text{so} \quad x = \det(A) \frac{1}{\det(A^T)\det(A)} \det(A^T) = 1 \ .$$

What is wrong with this argument?

(d) Show that x is always either 0 or 1.

(*Hint for part* (d): use the equality in part (b), look at the determinant of each side, and use the fact that determinant "respects" multiplication.)

Problem 4: What is the *volume* of the 3-dimensional parallelotope in $\mathbb{R}^4$ generated by

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad .$$