HW1 Solutions

Math 2310

1.1.1

The combinations give:

(a) a line in $\mathbb{R}^3$, since $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) the plane in $\mathbb{R}^3$ of points where the (y,z) part is a scalar multiple of (2,3) (and x is any value).

(c) all of $\mathbb{R}^3$. To see this, check that for any two of the three vectors, all of their linear combinations form a plane, and the third vector is outside that plane.

1.1.3

$(v + w) + (v - w) = 2v = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$, so $v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

$(v + w) - (v - w) = 2w = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$, so $v = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

1.1.5

$u + v + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $2u + 2v + w = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

Since $u + v + w = 0$, $w = -u - v$, so it is a linear combination of u and v.

1.1.13

(a) For every clock hour vector, like 1:00, its opposite vector is also on the clock, in this case 7:00, so adding all the clock vectors gives the zero vector since each one cancels out with its opposite.

(b) The sum, V, is 0, so removing 2:00 gives $V - 2:00$ and $-2:00 = 8:00$ on the clock.

(c) 2:00 is 30 degrees (or $\frac{\pi}{6}$ in radians) above horizontal (a third of a right angle), so if it is a unit vector its x, y components are $(\sqrt{3}/2, 1/2)$.

1.1.19

First draw v and w, then draw the ray going infinitely from zero in the direction they point. The linear combinations $cv + dw$ with non-negative c and d are the points in between these two rays (which do in fact look kind of like a cone).

1.1.26

We need to solve the equations

$c + 3d = 14$

$2c + d = 8$

We can solve these by substitution: $c = 14 - 3d$ so $(28 - 6d) + d = 8$, hence $20 = 5d$, so $d = 4$ and $c = 14 - 12 = 2$, but soon we will see an easier way to do this using matrices.

1.1.28

There are six unknown variables, the three in v and the three in w, and the two vector equations $v + w = (4, 5, 6)$ and $v - w = (2, 5, 8)$ are each made up of 3 separate equations in those variables, so there are also six equations.

We could solve this as a 6 variable, 6 equation linear system, but here it's faster to use $(v + w) + (v - w) = 2v = (6, 10, 14)$ and $(v + w) - (v - w) = 2w = (2, 0, -2)$ so $v = (3, 5, 7)$ and $w = (1, 0, -1)$.

1.1.31

The equations come from reading off each row of the equation $cu + dv + cw = b$

$2c - d = 1$

$-c + 2d - e = 0$

$-d + 2e = 0$

We then find that $d = 2e$, so now we need only solve $2c - 2e = 1$ and $-c + 3e = 0$. This gives us $c = 3e$, and $6e - 2e = 4e = 1$, so $e = \frac{1}{4}$, $d = 2e = \frac{1}{2}$, and $c = 3e = \frac{3}{4}$.

1.2.4

(a) -1. $v \cdot -v = cos\theta$. The angle θ between $v, -v$ is always π ($180°$), and $cos(\pi) = -1$.

(b) 0. Check that for any vectors v, w, x, y, $(v + w) \cdot (x + y) = v \cdot x + v \cdot y + w \cdot x + w \cdot y$. This means that $(v + w) \cdot (v - w) = v \cdot v - v \cdot w + w \cdot v - w \cdot w$. v and w are unit vectors, so $v \cdot v = w \cdot w = 1$, so we have $(v + w) \cdot (v - w) = w \cdot v - v \cdot w$. However, you can also check that $v \cdot w = w \cdot v$, so this is $(v + w) \cdot (v - w) = 0$.

(c) $(v - 2w) \cdot (v + 2w) = v \cdot v + 2v \cdot w - 2w \cdot v - 4w \cdot w = 1 + 2v \cdot w - 2v \cdot w - 4 = 1 - 4 = -3$.

1.2.6

(a) w and v are perpendicular if and only if $0 = w \cdot v = 2w_1 - w_2$, so w lies on the line $w_2 = 2w_1$ in the plane $\mathbb{R}^2$.

(b) Plane. $(x, y, z) \cdot (1, 1, 1) = 0$ when $x + y + z = 0$, which defines a plane in $\mathbb{R}^3$.

(c) Such a vector (x, y, z) has both $x + y + z = 0$ and $z + 2y + 3z = 0$. The planes defined by these two equations intersect in a line, and any vector perpendicular to both $(1, 1, 1)$ and $(1, 2, 3)$ is on that line.

1.2.8

(a) False. The vectors perpendicular to u form a plane as in problem 6, and any plane contains non-parallel vectors. In particular, both $(1, -1, 0)$ and $(0, 1, -1)$ are in that plane and they are not parallel.

(b) True. $u \cdot (v + 2w) = u \cdot v + 2u \cdot w = 0$.

(c) True. $||u - v|| = \sqrt{(u-v)\cdot(u-v)} = \sqrt{u \cdot u - 2u \cdot v + v \cdot v} = \sqrt{1 - 2 * 0 + 1} = \sqrt{2}$.

1.2.12

We want $0 = (w - cv) \cdot v = w \cdot v - cv \cdot v$, so $c = \frac{w \cdot v}{v \cdot v}$. When $v = (1, 1)$ and $w = (1, 5)$, $c = \frac{1+5}{1+1} = 3$.

1.2.16

$||v|| = \sqrt{v \cdot v} = \sqrt{1+1+1+1+1+1+1+1+1} = \sqrt{9} = 3$. $v/||v|| = (\frac{1}{3}, ..., \frac{1}{3})$ is then a unit vector in the same direction as v. For $w = (w_1, ..., w_9)$, $v \cdot w = 0$ means $w_1 + ... + w_9 = 0$, so, among others, $(1, -1, 0, ..., 0)$ is perpendicular to v and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, ..., 0)$ is a unit vector perpendicular to v.

1.2.27

$$
\begin{gathered}
||v + w||^2 + ||v - w||^2 \\
= (v + w) \cdot (v + w) + (v - w) \cdot (v - w) \\
= v \cdot v + v \cdot w + w \cdot v + w \cdot w \\
+ v \cdot v - v \cdot w - w \cdot v + w \cdot w \\
= 2v \cdot v + 2w \cdot w \\
= 2||v||^2 + 2||w||^2
\end{gathered}
$$