HW9 Solutions

Math 2310

6.3.4

$\frac{d(v+w)}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = w - v + v - w = 0$, so $v + w$ is constant.

$\frac{d\vec{u}}{dt} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{u}.$

The matrix A has rank 1, so 0 is an eigenvalue with $(1, 1)$ as its eigenvector, and since the trace is -2, the other eigenvalue is -2 with eigenvector $(1, -1)$.

Therefore the general solution is $\vec{u} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} + de^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and plugging in $\vec{u}(0) = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$ gives the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

with solution $c = 20$, $d = 10$. Therefore we have

$$\vec{u} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so

$$\vec{u}(1) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + e^{-2} \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

and

$$\vec{u}(\infty) = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

6.3.5

The eigenvalues are now 0 and +2. $e^{-\infty}$ goes to 0, but $e^{+\infty}$ is infinite, so $v(t)$ grows infinitely large over time.

6.3.8

We want the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$.

We have $0 = (6-\lambda)(1-\lambda)+4 = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5)$, so the eigenvalues are 2 and 5. The corresponding eigenvectors are $(1,2)$ and $(2,1)$ respectively. $r(0) = w(0) = 30$ gives the equation

$$\begin{bmatrix} r \\ w \end{bmatrix}(0) = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

with solution $(c,d) = (10,10)$. So at time t we have

$$\begin{bmatrix} r \\ w \end{bmatrix} = 10e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 10e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and for large t the second term dominates and the ratio of rabbits to wolves is 2 to 1.

6.3.11

$A^2 = 0$, so $A^n = 0$ for all $n \geq 2$, and therefore $e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

$e^{At}(y(0), y'(0)) = (y(0) + y'(0)t, y'(0))$, so the solution for $y(t)$ is $y(0) + y'(0)t$ with constant first derivative $y'(0)$.

6.3.13

(a) Both $asin(\pm 3t)$ and $bcos(\pm 3t)$ solve the equation. If we want $y'(0)$ to be 0 and $y(0)$ to be positive we should choose the cosine, and to make $y(0) = 3$ we get $b = 3$, so $y(t) = 3cos(3t)$.

(b) The eigenvalues of A solve $\lambda^2 + 9 = 0$, so $\lambda = 3i, -3i$ and the eigenvectors are $(1, 3i)$ and $(1, -3i)$. The initial conditions give the equations $c + d = 3$ and $3ic - 3id = 0$, so $c = d = 3/2$, therefore $y(t) = 3\frac{e^{3it}+e^{-3it}}{2} = 3cos(3t)$.

6.3.15

(a) Here $A = [1]$, the 1 by 1 identity matrix, so $u = 4$ is a particular solution. The homogeneous solution u_n to $\frac{du}{dt} = u$ is ce^t, so the complete solution is $ce^t + 4$.

(b) $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, so $A^{-1}b = (4, 2)$. Both eigenvalues of A are 1 but A is not diagonalizable so its only eigenvector is $(0, 1)$. Therefore the complete solution is $u(t) = (4, ce^t + 2)$.

6.3.21

The eigenvalues of A are $1, 0$ (as an upper triangular matrix) with eigenvectors $(1, 0)$ and $(4, -1)$. Therefore

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$$

and

$$e^{At} = Xe^{\Lambda t}X^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 4e^t \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$$

6.3.26

(a) Its inverse is e^{-At}. You can check that this works for the same reason e^{-x} is the inverse of e^x for real numbers.

(b) If $Ax = \lambda x$ then $e^{At}x = e^{\lambda}x$, and e^{λ} is always nonzero.

6.4.7

$$Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & -2 \end{bmatrix}$$

6.4.8

S is singular so 0 is an eigenvalue, and the trace is 25 so that must be the other eigenvalue. The corresponding eigenvectors are $(4, -3)$ and $(3, 4)$ which normalize to $\pm(\frac{4}{5}, -\frac{3}{5})$ and $\pm(\frac{3}{5}, \frac{4}{5})$. An orthogonal matrix diagonalizing S has as columns these eigenvectors in either order, each with either choice of sign, so there are 8 possible choices.

6.4.9

(a) Eigenvalues for this matrix solve $0 = (1-\lambda)^2 - b^2 = (1+b-\lambda)(1-b-\lambda)$, so $\lambda = 1 \pm b$. One of those is negative when $|b| > 1$.

(b) The second pivot of this matrix is $1 - b^2$, which is always negative if $|b| > 1$.

(c) $1 \pm b$ has, regardless of the sign of b, one value less than or equal to 1 and one value greater than or equal to 1, and the latter is always positive.

6.4.12

The fake proof assumes that the eigenvectors x are vectors of real numbers, which isn't always the case. Complex eigenvalues often have complex eigenvectors.

6.4.16

M is antisymmetric and also orthogonal, so its eigenvalues are pure imaginary and unit length, so all are either i or $-i$. As the trace is 0, the eigenvalues (with multipliciy) must be $i, i, -i, -i$.

6.4.20

1. They are perpendicular because when $S = S^T$ the nullspace is the same as the left nullspace which is orthogonal to the column space, which contains x.

2. If α and β are eigenvalues of A, then $S - \beta I$ has the same eigenvectors with eigenvalues shifted by β to get $\alpha - \beta$ and 0. So by the previous part the eigenspaces for these are

orthogonal, hence so are the eigenspaces for α and β.

6.4.23

(a) False. Any matrix with n distinct real eigenvalues has n real eigenvectors, so (for instance) any triangular (upper or lower) matrix with distinct diagonal entries is diagonalizable but not symmetric.

(b) True. The matrix then diagonalizes as $X\Lambda X^{-1}$ where $X^{-1} = X^T$. The transpose is then $(X\Lambda X^T)^T = (X^T)^T\Lambda^T X^T = X\Lambda X^T$, so it is symmetric.

(c) True. If A is invertible then $(A^T)^{-1} = (A^{-1})^T$, so if $A = A^T$ then $A^{-1} = (A^{-1})^T$. Also follows from the diagonalization as $X\Lambda^{-1}X^T$ is symmetric.

(d) False. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is symmetric with eigenvectors (1,1) and (1,-1), which can be arranged into a matrix as $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. They could also be put in the other order and be symmetric, but this arbitrariness is how we know X does not need to be symmetric.

6.4.35

(a) If $S = S^T$ and $S^T = S^{-1}$ then $S = S^{-1}$ so $SS = SS^{-1} = I$.

(b) The eigenvalues can only be ± 1.

(c) Λ is any diagonal matrix with only 1 or -1 in the diagonals.

6.5.2

S_1 is not positive definite as $ac - b^2 = 35 - 36 = -1 < 0$. For $x = (2, -1)$, $x^T S_1 x = 5 * 2 - 2 * 6 * 2 + 7 * 1 = 10 - 24 + 7 = -7 < 0$.

S_2 is not positive definite as $a = -1 < 0$.

S_3 is not positive definite as $ac - b^2 = 100 - 100 = 0$.

S_4 is positive definite as $a = 1 > 0$ and $ac - b^2 = 101 - 100 = 1 > 0$.

6.5.3 (last matrix)

Using the first test, we want $c > 0$ and $c^2 - b^2 > 0$ (so $|c| > |b|$).

$L = \begin{bmatrix} 1 & 1 \\ -b/c & 0 \end{bmatrix}$

$D = \begin{bmatrix} c & 0 \\ 0 & c - b/c \end{bmatrix}$

6.5.8

$S = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix}$, which indeed has pivots 3 and 4.

6.5.9

$S = 4\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$. Note that each entry in the matrix is a product of two terms in the expression $x_1 - x_2 + 2x_3$.

The only pivot is 4, the rank is 1, the eigenvalues are 0,0,24, the determinant is 0.

6.5.16

$(x_1, x_2, x_3) = (0, 1, 0)$, since $x^T Sx$ is then $s_{2,2}x_2^2 = 0$.

6.5.17

If a diagonal entry s_{jj} of a symetric matrix were smaller than all of the eigenvalues, then $S - s_{jj}I$ would have all positive eigenvalues and would then be positive definite. But $S - s_{jj}$ has a 0 on the main diagonal so it cannot be positive definite as seen in the previous problem.

6.5.30

To have a saddle point $ax^2 + 2bxy + cy^2$ needs to look like $\lambda_1 x^2 - \lambda_2 y^2$ with some rotation or reflection in the xy-plane, which is the same as the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ having one positive and one negative eigenvalue. This happens exactly when the determinant of the matrix is negative, so we want $b^2 > ac$.

6.5.36

(a) The eigenvalues of $\lambda_1 I - S$ are $0 \le \lambda_1 - \lambda_2 \le \cdots \le \lambda_1 - \lambda_n$, which are all nonnegative so $\lambda_1 I - S$ is positive semidefinite.

(b) $\lambda_1 I - S$ is positive semidefinite, so $x^T(\lambda_1 I - S)x \ge 0$, hence $\lambda_1 x^T x > x^T Sx$.

(c) Since $x^T x$ is always positive for $x \ne 0$, this gives the inequality $\lambda_1 \ge \frac{x^T Sx}{x^T x}$.