HW9 Solutions

Math 2310

2.4.35

In the 4 node circle graph, for each node there is an edge to the node on either side of it (where 1 and 4 are considered adjacent), so in the adjacency matrix there is a 1 on either side of each diagonal entry (wrapping around for 1 and 4). Note that this is different from the incidence matrices in 10.1 for directed graphs, since both dimensions of the matrix are indexed by the nodes.

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

This tells us that from each node there are two paths of length 2 to the opposite note (1 in either direction around the circle) and two paths of length 2 to itself (going out along 1 edge then back along the same edge in either direction around the circle).

10.1.1

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

The nullspace is spanned by $(1, 1, 1)$ and the row space is orthogonal to the nullspace (consists of vectors whose components sum to 0), so it cannot contain $(1, 0, 0)$.

10.1.4

If $x = (1,2,3)$, then $b = Ax = (1,2,1)$ has a solution to $AX = b$, namely x. $b = (1,0,0)$ has no solution. To be in the column space b must be orthogonal to $(1,-1,1)$ as that describes the only loop in the system (or alternatively, a basis vector for the left nullspace describing a uniform flow around the loop).

10.1.6

$$A^T A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

If $x = (1,2,3)$ then $b = A^T Ax = (-3,0,3)$ has a solution to $A^T Ax = b$, namely x.

10.1.8

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$x = (1,1,1,1,1)$ solves $Ax = 0$.

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

This row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so two solutions (which generate the nullspace of A^T) are $(1,-1,1,0,0)$ and $(-1,1,0,-1,1)$. (It will be convenient for the next problem to replace the second y with $(0,0,1,-1,1)$ which you get by adding the two y's above).

10.1.9

The conditions are $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ corresponding to the two loops in the graph and orthogonality with the two basis vectors for the left nullspace found in the previous problem. This is Kirchhoff's voltage law around the two loops in the graph.

10.1.10

Elimination gives the echelon form of A as

$$U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows are the incidence matrix for the subgraph containing edges (1,2,4), a spanning tree. The other spanning trees are edges (1,2,5),(1,3,4),(1,3,5),(1,4,5),(2,3,4),(2,3,5),(2,4,5).

10.1.11

$$A^T A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

(a) The diagonal entries give how many edges go into/out of each node.

(b) The off diagonals -1 or 0 tell which pairs of nodes are connected by an edge (-1 if they are, 0 if they're not).

10.1.12

(a) The nullspace of $A^T A$ and A are always the same and this is true for A.

(b) $A^T A$ is always positive semidefinite as $x^T A^T A x = (Ax)^T (Ax)$, the squared length of Ax which is always ≥ 0, but it cannot be positive definite as the nullspace is nontrivial so zero is an eigenvalue.

(c) The eigenvalues are real as $A^T A$ is symmetric, since $(A^T A)^T = A^T (A^T)^T = A^T A$. Their signs are all positive (or 0) because $A^T A$ is positive semidefinite.

10.1.15

We know that for l the number of loops, $n - m + l = 1$, so when $n = m = 7$, $l = 1$.

10.1.18

The complete graph with $n = 6$ nodes has exactly 1 edge for every pair of 2 nodes, so the number of edges is the number of pairs of nodes, which is 6 choose 2, computed as $6!/((6-2)!2!) = 6 * 5/2 = 15$. This formula comes from the idea that to find a pair of two nodes, you need to pick a first node from 6 choices and pick a second node from the remaining 5 choices, but we don't care about the order of the two nodes so divide by 2.

10.3.1

We know 1 is an eigenvalue so to get the right trace we have $1 + \lambda_2 = .9 + .85 = 1.75$, so $\lambda_2 = .75$.

The steady state eigenvector is the basis vector for the nullspace of $A - I$ whose entries sum to 1, which is $(.6, .4)$.

10.3.2

The other eigenvector is $(-1, 1)$ so $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$,

and $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$

10.3.3 (last matrix)

$$A-\lambda I=\tfrac{1}{4}\begin{bmatrix}2-4\lambda & 1 & 1\\ 1 & 2-4\lambda & 1\\ 1 & 1 & 2-4\lambda\end{bmatrix}\text{which has determinant}$$

$$(2-4\lambda)^3+1+1-(2-4\lambda)-(2-4\lambda)-(2-4\lambda)=-64\lambda^3+96\lambda^2-48\lambda+8+2-6+12\lambda=-64\lambda^3+96\lambda^2-36\lambda+4$$

Factoring cubic polynomials is hard, but it's easier when we already know one of the eigenvalues should be 1, so it should have a factor of $\lambda-1$. We can then find the remaining factor using polynomial long division which gives

$$-64\lambda^3+96\lambda^2-36\lambda+4=(\lambda-1)(-64\lambda^2+32\lambda-4)=-64(\lambda-1)(\lambda^2-\frac{1}{2}\lambda+\frac{1}{16})=-64(\lambda-1)(\lambda-\frac{1}{4})^2$$

so the eigenvalues are $1, \frac{1}{4}, \frac{1}{4}$, and the steady state eigenvector is the basis element of the nullspace of $A-I$ whose entries sum to 1, which is $\frac{1}{3}(1,1,1)$.

10.3.5

The steady state is $(0,0,1)$, which means everyone is dead. This makes sense since in this (rather morbid) model people age and die but no new people are born.

10.3.7

Since A is a markov matrix with trace 1.5, its eigenvalues are 1 and .5. The eigenvectors are (.6,.4) and (-1,1). The diagonalization gives

$$A=\begin{bmatrix}.6 & -1\\ .4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & .5\end{bmatrix}\begin{bmatrix}1 & 1\\ -.4 & .6\end{bmatrix}$$

which gives us

$$A^\infty=\begin{bmatrix}.6 & -1\\ .4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -.4 & .6\end{bmatrix}=\begin{bmatrix}.6 & .6\\ .4 & .4\end{bmatrix}$$

This result for A^∞ relies on the top eigenvalue being 1, the other eigenvalue having $|\lambda_2|<1$, and the first eigenvector being $(.6,.4)$. Furthermore since both columns in A add up to 1 it must be the case that the differences between entries in each row are the same, so $(-1, 1)$ will always be the other eigenvector. Therefore the Markov matrices A with a steady state of $(.6, .4)$ are given by

$$\begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 + .4\lambda_2 & .6 - .6\lambda_2 \\ .4 - .4\lambda_2 & .4 + .6\lambda_2 \end{bmatrix}$$

The above matrices with $-1 < \lambda_2 < 1$ and all entries between 0 and 1 (so $\lambda_2 > -.4/.6$) are all of the Markov matrices with this steady state.

10.3.9

M^2 will always have positive entries, so we only need to show that the columns still add up to 1. Columns adding up to 1 is the same as the equation $[1\ 1\ \cdots\ 1\ 1]M = [1\ 1\ \cdots\ 1\ 1]$, and if it holds, then $[1\ 1\ \cdots\ 1\ 1]MM = [1\ 1\ \cdots\ 1\ 1]M = [1\ 1\ \cdots\ 1\ 1]$ so M^2 has the same property.

10.3.10

Its eigenvalues are 1 and $a + d - 1$ and the steady state eigenvector is $\frac{1}{b+1-a}(b, 1 - a)$

10.4.1

The feasible set is the line segment from (6,0) to (0,3), where those endpoints are found by setting each variable in the constraint equation to 0. The cost is minimized at (6,0) and maximized at (0,3).

10.4.3

The corners of the set satisfying $x_1 + 2x_2 - x_3 = 4$ in $\mathbb{R}^3$ are $(4, 0, 0)$, $(0, 2, 0)$, and $(0, 0, -4)$ which is not in the feasible set as it has a negative value. Seen in the plane, the feasible set contains the segment from (4,0,0) to (0,2,0) and extends infinitely in the positive directions ($x_3 = x_1 + 2x_2 - 4$ can all get arbitrarily large hence so can the cost).

(The problem says there is no minimum but if the cost is assumed to be positive it's really that there's no maximum.)

10.4.5

If $x_1 + x_2 + 2x_3 = 4$ and the cost is given by $5x_1 + 3x_2 + 7x_3$, then starting from the corner $(4, 0, 0)$ the cost r saved by moving to $(3, 1, 0)$ is 2 (computed as the difference 20-18) and the cost r saved by moving to $(2, 0, 1)$ is 3 (computed as the difference 20-17). Therefore the simplex method moves to the corner $(0, 0, 2)$ with cost 14. From there, moving an hour back to the PhD would add cost, but giving the student an hour reduces cost as (0,1,1.5) has cost 13.5, so the simplex finally finds the minimum solution (0,4,0) with cost 12.

10.4.6

If the cost vector is [2 3 7] then the PhD gets the job as their total cost is 8, below 12 for the student and 14 for the machine. This is unrealistic though, as any self respecting PhD would charge more than the student even if their pace is the same.

In the dual problem, the cheater is competing with the PhD, student, and machine, so their rate y per problem can't exceed those three, hence $y \leq 2$, $y \leq 3$, and $2y \leq 7$ (y must also be positive). Their goal is to maximize income for doing 4 problems, so the problem is to maximize 4y.

10.4.7

To find the other corners, set $x_1 = 0$ to see that $x_2 + 2x_3 = 4$ and $x_2 + x_3 = 6$ has solution (8,-2), outside the admissible set, and setting $x_2 = 0$ gives $x_1 + 2x_3 = 4$ and $2x_1 + x_3 = 6$ with solution $(8/3, 2/3)$). The feasible set is then the line between $(2, 2, 0)$ and $(8/3, 2/3)$ and $(2, 2, 0)$ has lower cost at 16 compared to $56/3 \approx 18.66$.