HW2 Solutions

Math 2310

2.3.1

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

(c) To take a vector v and first exchange rows 1 and 2 we take $P_{12}v$, then to further exchange rows 2 and 3 we take $P_{23}(P_{12}v) = (P_{23}P_{12})v$, so we are looking for the product

$$P_{23}P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2.3.3

The first two are clear from looking at A:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

For E_{32} we need to look at

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

which tells us that

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Now we get

$$M = E_{32}E_{31}E_{21} = E_{32}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

2.3.8

$M^* = \begin{bmatrix} a & b \\ c - la & d - lb \end{bmatrix}$ and $detM^* = a(d - lb) - b(c - la) = ad - alb - bc + bla = ad - bc.$

2.3.19

$$PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $M^2 = I$.

2.3.27

(a) If $d = 0$ and $c = 1$ there is no solution (for any a, b) as the bottom equation in the system is then $0x + 0y + 0z = 1$, which is never true.

(b) If $d = 0$ and $c = 0$ there are infinitely many solutions (for any a, b), since the bottom equation in the system is $0 = 0$, which doesn't add any new information. Since the left sides of the first two equations are not multiples of each other (because the same is true of the non-augmented part of the matrix rows), the first and second equations each have a plane of solutions which intersect in a line, containing infinitely many solutions to the system.

a and b have no effect here on solvability.

2.3.31

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

$$E_{43}E_{32}E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

2.4.1

$$BA = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$$

$$ABD = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

DC is not allowed as C has 1 column and D has 3 rows.

$A(B+C)$ is not allowed as $B+C$ does not make sense since B and C have different dimensions.

2.4.3

$$AB + AC = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

They are the same! In fact it is always the case that $A(B+C) = AB + AC$.

2.4.13

$AB = BA$ means that $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ so $b = 0$ and $c = 0$.

$AC = CA$ means that $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ so $c = 0$ and $a = d$.

2.4.14

$(A-B)^2$ is always equal to $(B-A)^2$, $A(A-B)-B(A-B)$, and $A^2-AB-BA+B^2$.

It is not equal to A^2-B^2 unless $-AB-BA=0$, which is not generally true, and is not equal to $A^2-2AB+B^2$ unless $AB=BA$, which is also not generally the case.

2.4.32

$AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, because A acts independently on each column of $[x_1 x_2 x_3]$ to give $[Ax_1 Ax_2 Ax_3]$.

2.4.34

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (a+b) & (a+b) \\ (c+d) & (c+d) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a+c) & (b+d) \\ (a+c) & (b+d) \end{bmatrix}$$

For these to be equal, we need:

$a+b=a+c$, so $b=c$,

$a+b=b+d$, so $a=d$,

$c+d=a+c$, so $a=d$, and

$c+d=b+d$, so $b=c$. Therefore A must be of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

2.5.2

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.5.5

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.5.7ab

(a) $Ax = (0, 0, 1)$ has no solution because (row 1)$\cdot x = 0$ and (row 2)$\cdot x = 0$, so

(row 3)$\cdot x =$(row 1 + row 2)$\cdot x =$(row 1)$\cdot x+$(row 2)$\cdot x = 0 + 0 = 0$,

which contradicts (row 3)$\cdot x = 1$.

(b) There can only be a solution if $b_1 + b_2 = b_3$.

2.5.11

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2.5.12

$A^{-1} = BC^{-1}$

2.5.23

First eliminate below the pivots:

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

R2 - 1/2 R1: $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$

R3 - 2/3 R2: $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix}$

Next eliminate above the pivots:

R2 - 3/4 R3: $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix}$

R1 - 2/3 R2: $\begin{bmatrix} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix}$

1/2 R1, 2/3 R2, 3/4 R3: $\begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{bmatrix}$

2.5.26

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$E_{21}A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$E_{12}E_{21}A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$D^{-1}E_{12}E_{21}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = D^{-1}E_{12}E_{21}A = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

2.5.27

For the first matrix $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

For the second $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

2.5.28

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$$

2.5.32

If $x = (c, c, ..., c)$ for any $c \in \mathbb{R}$, then any permutation matrix does not change x, so $(P - Q)x = Px - Qx = x - x = 0$. But there are infinitely many vectors x of this form

(1 for every choice of c), so $(P-Q)x=0$ does not have a unique solution, hence $P-Q$ is singular.

2.5.33

We can do the entire elimination process for block matrices, but it only works when the matrices inverted below are actually invertible. Also note that the order of multiplication matters

$$\begin{bmatrix} I & 0 & I & 0 \\ C & I & 0 & I \end{bmatrix} \to \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & -C & I \end{bmatrix}$$

$$\begin{bmatrix} A & 0 & I & 0 \\ C & D & 0 & I \end{bmatrix} \to \begin{bmatrix} A & 0 & I & 0 \\ 0 & D & -CA^{-1} & I \end{bmatrix} \to \begin{bmatrix} I & 0 & A^{-1} & 0 \\ 0 & I & -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & I & I & 0 \\ I & D & 0 & I \end{bmatrix} \to \begin{bmatrix} I & D & 0 & I \\ 0 & I & I & 0 \end{bmatrix} \to \begin{bmatrix} I & 0 & -D & I \\ 0 & I & I & 0 \end{bmatrix}$$