HW2 Solutions

Math 2310

2.6.5

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$U = EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

2.6.7

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

2.6.9

Some textbooks seem to have different versions of this problem, which will not be graded for correctness.

2.6.12

Some textbooks seem to have different versions of this problem, which will not be graded for correctness.

2.7.16

(a) $(AA)^T = A^T A^T = AA$ so A^2 and likewise B^2 are symmetric, hence their difference is also symmetric.

(b) $(A + B)(A - B) = A^2 + BA - AB - B^2$, which by (a) is symmetric if and only if $BA - AB$ is symmetric. $(BA - AB)^T = A^T B^T - B^T A^T = AB - BA = -1(BA - AB)$, so if AB is nonzero this is NOT symmetric.

(c) $(ABA)^T = A^T B^T A^T = ABA$ since transpose reverses multiplication order.

(d) $(ABAB)^T = B^T A^T B^T A^T = BABA$ which is not generally the same as $ABAB$.

2.7.17ab

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2.7.20

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

2.7.39

$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ and $Q^T = \begin{bmatrix} q_1^T \\ \cdots \\ q_n^T \end{bmatrix}$, so $(Q^TQ)_{ij} = q_i^T q_j = q_i \cdot q_j$ and since $Q^TQ = I$, this means $q_i \cdot q_j = 1$ when $i = j$ and $q_i \cdot q_j = 0$ when $i \neq j$.

(a) $q_i \cdot q_i = 1$ so each q_i is a unit vector.

(b) For $i \neq j$, $q_i \cdot q_j = 0$ so q_i and q_j are perpendicular.

(c) We want the columns to be unit vectors so if $A = \begin{bmatrix} cos\theta & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$, then $cos^2\theta + q_{21}^2 = 1$. So a good choice for q_{21} is $sin\theta$ ($-sin\theta$ would also work). So the column q_1 is just the point with angle θ on the unit circle, and since q_2 must be perpendicular to it we can use $theta + \frac{\pi}{2}$ (or $\theta - \frac{\pi}{2}$). We then get

$$Q = \begin{bmatrix} cos\theta & cos(\theta + \frac{pi}{2}) \\ sin\theta & sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Checking this has $Q^TQ = I$ gives

$$\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} cos^\theta + sin^2\theta & -cos\theta sin\theta + sin\theta cos\theta \\ -sin\theta cos\theta + cos\theta sin\theta & sin^2\theta + cos^2\theta \end{bmatrix} =$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as desired.

3.1.1

Conditions (1), (2), and (8) don't hold here.

3.1.3

(a) With the usual scaling, this space is only closed under scaling cx with c is positive, which isn't allowed. Also, (3) is broken as there is no zero vector, (4) is broken as there is no $(-x) > 0$ if $x > 0$, and (6),(7),(8) don't really make sense as not all scalings are positive numbers.

(b) $c(x+y)$ is the usual $(xy)^c$, while $cx + cy$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(2+1) = 8$. The zero vector is the number 1.

3.1.6

$3(x^2) - 4(5x) = 3x^2 - 20x$

3.1.15

(a) The intersection of two planes through $(0,0,0)$ in $\mathbb{R}^3$ is probably a *line* in $\mathbb{R}^3$ but it could be a *plane*.

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a *point* but it could be a *line*.

(c) If x, y are in $S \cap T$ then they are in S and in T, so $cx + dy$ is in S and is in T for all $c, d \in \mathbb{R}$, so $cx + dy$ is therefore in $S \cap T$, which is therefore a subspace.

3.1.25

$z = x + y$.

3.1.27

(a) False. 0 is always in the column space, so the vectors not in the column space don't include 0.

(b) True. If A has any nonzero entry then the column containing that entry is a nonzero vector in the column space.

(c) True. The span of any set of vectors is the same as the span of $2*$ each of those vectors.

(d) False. If $A = I$ the column space of A is all of R^n and the column space of $A - I = I - I = 0$ is the zero vector.

3.1.29

If the 9 by 12 system $Ax = b$ is solvable for every b, then $C(A) = \mathbb{R}^9$. As every vector b in $\mathbb{R}^9$ is a linear combination of the columns of A.

3.2.1

(a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ with free variables x_2, x_4, x_5 and pivot variables x_1, x_3.

(b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ with free variable x_3 and pivot variables x_1, x_2.

3.2.2

(a) (-2,1,0,0,0), (0,0,-2,1,0), (0,0,-3,0,1)

(b) (1,-1,1)

3.2.3

(a) $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

R has the same nullspace as U since they are related by row operations (which are invertible and don't affect the zero vector).

3.2.4

(a) (3,1,0) and (5,0,1)

(b) (3,1,0)

The number of pivots plus free variables is n, since every column either has a pivot or acts on a free variable.

3.2.6

(a) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3.2.16

$A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ has rank 3.

3.3.1

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}$$

$Ax = b$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of (2, 2, 2) and (4, 5, 3). This is the plane $b_3 + b_2 - 2b_1 = 0$. The nullspace contains all combinations of $s_1 = (-1, -1, 1, 0)$ and $s_2 = (2, -2, 0, 1)$; $x_{complete} = x_p + c_1 s_1 + c_2 s_2$.

$$[Rd] = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{gives the particular solution } x_p = (4, -1, 0, 0).$$

3.3.5

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} \text{is solvable when } b_3 - 2b_1 - b_2 = 0.$$

Back-substitution gives the particular solution to $Ax = b$ and the special solution to $ax = 0$:

$$x = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

3.3.23

For A, $q = 3$ gives rank 1, every other q gives rank 2. For B, $q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

3.3.34

(a) If $s = (2, 3, 1, 0)$ is the only special solution to $Ax = 0$, the complete solution is $x = cs$ (a line of solutions). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is not free in s, and R must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $Ax = b$ can be solved for all b, because A and R have full row rank $r = 3$.

3.3.36

Yes! Let b be the first column of A. Then $x = (1, 0, 0, ...)$ is a solution to $Ax = b$ so x is also a solution to $Cx = b$, which means b is also the first column of C, so A and C have the same first column. This can be repeated for each column, so A and C have all the same columns and are therefore equal.