HW5 Solutions

Math 2310

3.4.4

If a, d, f are all nonzero, then f can be used to eliminate e, c and d can then be used to eliminate b. Then each row can be scaled to have 1 on the diagonal, yielding the identity. Therefore the matrix can be reduced to the identity by row operations, so the only solution to $Ux = 0$ is 0.

3.4.5

Using row reduction we find that

(a) The matrix with these vectors as rows reduces to an upper triangular with all three pivots, so they are independent.

(b) The matrix with these vectors as rows reduces to a matrix with the bottom row all 0s, so they are dependent (in particular we can see that the third is -1 times the sum of the first two)

3.4.7

$v_2 - v_1 = w_1 - w_3 - w_2 + w_3 = w_1 - w_2 = v_3$, so v_3 is a linear combination of v_1, v_2. Therefore $-v_1 + v_2 - v_3 = 0$.

$[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3]A$ where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, and this matrix A is easily checked by elimination to be singular.

3.4.13

A row reduces to U, so the row spaces of A and U are the same, and the column spaces of A and U have the same dimensions. In U it is clear that the row space has dimension

2, hence so does the column space (as the second column is clearly a linear combinatino of the first and third).

(a) 2 (b) 2 (c) 2 (d) 2

3.4.23

For A, row 3 is the same as row 1 and the first two rows are clearly independent (as row 2 is 0 in the first component and row 1 is not), so the first two rows form a basis. U clearly has the same basis for its row space, so elimination fixes the row space.

For A column 2 is the sum of columns 1 and 3, so as we know from the row space that $C(A)$ is 2-dimensional, $\{(1,0,1),(2,1,2)\}$ forms a basis. For U we also have column 2 as the sum of columns 1 and 3, but this basis is now $\{(1,0,0),(2,1,0)\}$. These spaces have the same dimension but the column space of U is precisely the xy-plane in $\mathbb{R}^3$ while that of A includes vectors with a z-component, so elimination does not preserve the column space, only its dimension.

We know that elimination preserves the null space, as we've been using that fact for a while now to solve equations, so we only need to find the nullspace of U, which is $x_3(1,-1,1)$, or equivalently $span\{(1,-1,1)\}$.

3.4.30

The space of 2×3 matrices with $(2,1,1)$ in the nullspace is the space of matrices A of the form $[v_1\ v_2\ v_3]$ for column vectors v_1, v_2, v_3 such that $2v_1+v_2+v_3=0$. Therefore, v_1 and v_2 can be anything and $v_3=-2v_1-v_2$, so a basis for this space is the same as a basis for pairs of vectors v_1, v_2 with v_3 computed appropriately. This looks like $\{A_{00}, A_{01}, A_{1,0}, A_{1,1}\}$, where

$$A_{00}=\begin{bmatrix}1&0&-2\\0&0&0\end{bmatrix}, A_{01}=\begin{bmatrix}0&1&-1\\0&0&0\end{bmatrix}, A_{10}=\begin{bmatrix}0&0&0\\1&0&-2\end{bmatrix}, \text{and } A_{11}=\begin{bmatrix}0&0&0\\0&1&-1\end{bmatrix}.$$

3.5.1

(a) The dimension of the row and column spaces are the same as the rank: 5. The dimension of the null space is the number of columns minus the rank, here 9-5=4. The dimension of the left null space is the number of rown minus the rank, here 7-5=2.

(b) If the matrix has rank 3 then its column space is 3-dimensional, hence all of $\mathbb{R}^3$. As there are 3 rows and the rowspace has dimension 3, the left nullspace is $\{\vec{0}\} \subset \mathbb{R}^3$.

3.5.3

Row space: $\{(0,1,2,3,4),(0,0,0,1,2)\}$. Column space: $\{(1,1,0),(3,4,1)\}$. Null space: $\{(1,0,0,0,0),(0,2,-1,0,0),(0,2,0,-2,1)\}$. Left null space: $\{(1,-1,1)\}$.

3.5.4

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Both desired column space vectors are in fact columns, and the first and third rows span all of $\mathbb{R}^2$ which of course contains both desired row space vectors.

(b) This is impossible as the number of columns is 2: the dimension of the column space, 1, plus the dimension of the null space, 1. But if vectors in the null space are 3-dimensional as is the desired basis vector, this is a contradiction as then there must be 3 columns.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The rank is 3, so the null space is 0-dimensional and the left nullspace is 1-dimensional.

(d) $\begin{bmatrix} -9 & 3 \\ -3 & 1 \end{bmatrix}$.

(e) This is impossible because if the row and column spaces are the same the matrix must be square, so the number of rown minus the rank is the same as the number of columns minus the rank. This means the null space and left null space have the same dimension. In the next section you see that the null space is orthogonal to the row space and the left null space is orthogonal to the column space, so if the row space and column space are the same so must be the null space and left null space.

3.5.8

A: row space and column space are 3 dimensional. Null space is 2 dimensional. Left null space is 0 dimensional.

B: row space and column space are 3 dimensional. Null space is 3 dimensional. Left null space is 2 dimensional.

C: row space and column space are 0 dimensional. Null space is 2 dimensional. Left null space is 3 dimensional.

3.5.11

(a) $r < m$ as there must be dependent rows. $r \leq n$ as the rank never exceeds the number of columns. There is no guaranteed inequality between m and n.

(b) The rank is less than the number of rows so the left null space has positive dimension.

3.5.12

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. By definition of the null space, the dot product of any vector x in the null space with any row vector is 0, so dot product of x with any linear combination of row vectors is also 0. Therefore if any vector were in both the row space and the null space its dot product with itself would be 0, which can't happen for a nonzero vector.

3.5.15

Row exchange does not affect the row space (it's still the span of the same vectors) or the null space (this is why row operations help us solve linear equations). The column space and left null space change. The new left null space contains $(2, 1, 3, 4)$.

3.5.21

$$A = \begin{bmatrix} u_1v_1 + w_1z_1 & \cdots & u_1v_n + w_1z_n \\ \vdots & & \vdots \\ u_mv_1 + w_mz_1 & \cdots & u_mv_n + w_mz_n \end{bmatrix}$$

(a) u and w span the column space as each column vector is a linear combination of them.

(b) v and z span the row space as each row vector is a linear combination of them.

(c) The rank is less than 2 if u, w are dependent or if v, z are dependent.

(d) Here the rank is 2.

4.1.4

If $AB = 0$ then the columns of B are in the *null space* of A. The rows of A are in the *left null space* of B. With $AB = 0$, The left null space of A has dimension at least that of the column space of B, here 2. But then A cannot have rank 2, as then the dimension of the row space is 2 and the number of rows is the sum of the dimensions of the row space and left null space which would then be 4 not 3.

4.1.5

(a) $y^T b = 0$ because $y^T Ax = y^T b = 0$ (for x some such solution) since $y^T A = 0$.

(b) $y^T Ax = 0$ for such a y and $y^T A = [1\ 1\ 1]$, so $(1, 1, 1) \cdot x = 0$.

4.1.6

$y_1 = 1, y_2 = 1, y_3 = -1$. $(1, 1, -1)$ is in the left null space of this matrix.

4.1.9

If $A^T Ax = 0$ then $Ax = 0$. Reason: Ax is in the *nullspace* of A^T and also in the *column space* of A and those spaces are orthogonal.

4.1.19

Suppose L is a one-dimensional subspace (a line) in $\mathbb{R}^3$. Its orthogonal complement $L^\perp$ is the *plane* perpendicular to L. Then $(L^\perp)^\perp$ is a *line* perpendicular to $L^\perp$. In fact $(L^\perp)^\perp$ is the same as L.

4.1.24

The first column in A^{-1} is orthogonal to the space spanned by all but the first row of A.

4.1.30

$dim(N(A)) \geq dim(C(B)) = rank(B)$. By rank-nullity since A has 4 columns, $4 = rank(A) + dim(N(A)) \geq rank(A) + rank(B)$.