HW6 Solutions

Math 2310

4.2.1

(a) $p = \frac{a^Tb}{a^Ta}\vec{a} = \frac{1+2+2}{1+1+1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$e = b - p = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ and $e \cdot a = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$.

(b) $p = \frac{a^Tb}{a^Ta}\vec{a} = \frac{-1-9-1}{1+9+1}\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

$e = b - p = 0$ so $e \cdot a = 0$. In this case b is already on the line through a so projection doesn't change anything.

4.2.2

(a) The projection of $b = (cos\theta, sin\theta)$ onto $a = (1, 0)$ is $p = (cos\theta, 0)$

(b) The projection of $b = (1, 1)$ onto $a = (1, -1)$ is $p = (0, 0)$ since $a^Tb = 0$.

The picture for part (a) has the vector b at an angle θ with the horizontal a.

The picture for part (b) has vectors a and b at a 90 degree angle.

4.2.5

$a_1a_1^T/a_1^Ta_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$.

$$a_2a_2^T/a_2^Ta_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

$$\frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

because a_1 and a_2 are perpendicular so projecting onto one and then the other always yields 0.

4.2.6

$$a_1a_1^T/a_1^Ta_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

$$a_2a_2^T/a_2^Ta_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}.$$

$$a_3a_3^T/a_3^Ta_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}.$$

$$\frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

4.2.7

$$P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

$$P_1+P_2+P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4.2.11

(a) $p = A\hat{x} = A(A^TA)^{-1}A^Tb = (2,3,0)$, $e = b - p = (0,0,4)$, and $A^Te = 0$.

(b) $p = A\hat{x} = A(A^TA)^{-1}A^Tb = (4,4,6)$, $e = 0$ because b is in the column space of A.

4.2.12

(a) $P_1 = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $P_2 = A(A^TA)^{-1}A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4.2.13

P is square, given by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

4.2.17

$(I-P)^2 = I^2 - IP - PI + P^2 = I - P - P + P = I - P$. $(I-P)b = b - Pb = e$, so P projects onto the space of vectors perpendicular to the column space of A, which happens to be the left nullspace of A.

4.2.23

When A is invertible, the column space of A is the entire space $\mathbb{R}^n$ in which it lives, so any vector b in $\mathbb{R}^n$ is already in that column space and the projection doesn't do anything to it. The error e is 0.

4.2.24

The nullspace of A^T is *orthogonal* to the column space $C(A)$. So if $A^Tb = 0$, the projection of b onto $C(A)$ should be $p = 0$. $Pb = A(A^TA)^{-1}A^Tb = A(A^TA)^{-1}0 = 0$.

4.2.27

The vector Ax is in the nullspace of A^T. Ax is always in the column space of A. To be in both of those perpendicular spaces, Ax must be 0.

4.2.30

(a) The columns of the matrix are not independent, so we must replace A with the vector $a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ which has the same column space but independent columns.

$$P_C = a(a^Ta)^{-1}a^T = \tfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}.$$

(b) The row space is spanned by $a = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$, so

$$P_R = a(a^Ta)^{-1}a^T = \tfrac{1}{81}\begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix} = \tfrac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

$$P_CAP_R = \tfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}\tfrac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} = A.$$

This makes sense because P_CA projects the columns of A onto the column space of A which they are already in, and $AP_R = A$ as P_R is the projection onto the column space of A^T so $P_RA^T = A^T$ so $AP_R^T = AP_R = A$.

4.2.31

Check that $a_i \cdot (b-p) = 0$ for all i. If so then $b-p$ is perpendicular to the subspace spanned by $a_1, ..., a_n$ and hence p is the projection of b onto that subspace.

4.3.1

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \text{ give } A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \text{ and } A^T b = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

$$A^T A\hat{x} = A^T b \text{ gives } \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \text{ and } e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}.$$

$E = ||e||^2 = 44$.

4.3.2

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}. \text{ This } Ax = b \text{ is unsolvable.}$$

$$\text{Project } b \text{ to } p = Pb = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}. \text{ When } p \text{ replaces } b,\ \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ exactly solves } A\hat{x} = p.$$

4.3.3

In problem 2, $p = A(A^T A)^{-1} A^T b = (1, 5, 13, 17)$ and $e = b - p = (-1, 3, -5, 3)$. This e is perpendicular to both columns of A. This shortest distance $||e||$ is $\sqrt{44}$.

4.3.5

$E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $AT = [1\ 1\ 1\ 1]$ and $ATA = [4]$. $ATb = [36]$ and $(A^TA)^{-1}A^Tb = 9 =$ best height C for the horizontal line. Errors $e = b - p = (-9, -1, -1, 11)$ still add to zero.

4.3.9

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 20 \end{bmatrix} \text{ is unsolvable.}$$

$$A^TA\hat{x} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}.$$

4.3.10

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. This has solution $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$, so $p = b$ and $e = 0$ as b was already in the column space.

4.3.17

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\hat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

4.3.18

$p = A\hat{x} = (5, 13, 17)$ gives the heights of the closest line. The vertical errors are $b - p = (2, -6, 4)$. This error e has $Pe = Pb - Pp = p - p = 0$.

4.3.26

The unsolvable equations for $C + Dx + Ey = (0, 1, 3, 4)$ at the 4 corners are

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Then $A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^Tb = \begin{bmatrix} 8 \\ -3 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -3/2 \\ -3/2 \end{bmatrix}$.

At $(x, y) = (0, 0)$ the best plane $2 - \frac{3}{2}x - \frac{3}{2}y$ has height $C = 2 =$ average of 0,1,3,4.