HW7 Solutions

Math 2310

4.4.1

(a) Independent, change the second vector to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Orthogonal, change the second vector to $\begin{bmatrix} .8 \\ -.6 \end{bmatrix}$.

(c) Orthonormal.

4.4.2

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

$$Q^TQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}.$$

4.4.3

(a) A^TA is the 3×3 diagonal matrix with the columns of A's lengths squared on the diagonal.

(b) A^TA is the 3×3 diagonal matrix with $1, 4, 9$ on the diagonal.

4.4.4

(a) From 4.4.2, $Q = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}$.

(b) Any vector and 0 are orthogonal (dot product is 0) but not linearly independent.

(c) For this pick two other vectors independent of $q_1 = (1,1,1)/\sqrt{3}$ and apply Gram-Schmidt, or alternatively find two vectors orthogonal to q_1 (and each other) and normalize them. For example, $q_2 = (1,-1,0)/\sqrt{2}$ and $q_3 = (1,1,-2)/\sqrt{6}$.

4.4.7

The solution $\hat{x}$ satisfies $Q^TQ\hat{x} = Q^Tb$ but $Q^TQ = I$ so $\hat{x} = Q^Tb$.

4.4.14

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\sqrt{2}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

4.4.15

(a) Gram-Schmidt yields $q_1 = \frac{1}{3}(1,2,-2)$, $q_2 = \frac{1}{3}(2,1,2)$, and $q_3 = \frac{1}{3}(2,-2,-1)$.

(b) The left nullspace.

(c) $\hat{x} = (A^TA)^{-1}A^T(1,2,7) = (1,2)$.

4.4.18

$q_1 = \frac{1}{\sqrt{2}}(1,-1,0,0)$, $q_2 = \frac{1}{\sqrt{6}}(1,1,-2,0)$, $q_3 = \frac{1}{\sqrt{12}}(1,1,1,-3)$.

4.4.22

$A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. Completing Gram-Schmidt would also divide these by their lengths.

4.4.24

(a) One basis for A is the 3 special solutions $(-1,1,0,0),(-1,0,1,0),(1,0,0,1)$ to the equation $x_1 + x_2 + x_3 - x_4 = 0$.

(b) S is the subspace of vectors perpendicular to the vector $(1,1,1,-1)$, so $(1,1,1,-1)$ is a basis for $S^{\perp}$.

(c) Project onto both subspaces $S, S^{\perp}$: $b_1 = \frac{1}{2}(1,1,1,3)$ and $b_2 = \frac{1}{2}(1,1,1,-1)$.

4.5.1

$det(2A) = 2^4 det(A) = 16/2 = 8$, as by linearity each of the 4 rows multiplied by 2 contributes a factor of 2.

$det(-A) = (-1)^4 det(A) = 1/2$ for the same reason.

$det(A^2) = det(A)det(A) = 1/4$.

$det(A^{-1})det(A) = det(A^{-1}A) = det(I) = 1$, so $det(A^{-1}) = 1/det(A) = 2$.

4.5.3

(a) False. For I 2×2, $det(I + I) = det(2I) = 2^2 det(I) = 4det(I) \neq det(I) + det(I)$.

(b) True, by repeated application of $|AB| = |A||B|$.

(c) False. If A is $n \times n$ for $n > 1$ then $det(4A) = 4^n det(A) \neq 4det(A)$.

(d) False. For $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is invertible.

4.5.4

For J_3, exchange rows 1 and 3 to get I. which shows that $det(J_3) = -1$.

For J_4, exchange rows 1 and 4 then 2 and 3 to get I, which shows that $det(J_4) = (-1)^2 = 1$.

4.5.8a

$1 = det(I) = det(Q^T Q) = det(Q^T)det(Q) = det(Q)det(Q) = det(Q)^2$, and the only possible determinants $det(Q)$ which square to 1 are 1 or -1.

4.5.10

If each row of A sums to 0 then $A(1, ..., 1) = 0$ so A has nontrivial nullspace and is not invertible, so $det(A) = 0$.

If each row sums to 1 then $A - I$ has each row sum to 0, so $det(A - I) = 0$. $det(A)$ is not necessarily 1 as the matrix $\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$ has this property but its determinant is 13.

4.5.11

$|C||D| = (-1)^n |D||C|$ where the matrices are $n \times n$. Therefore this statement is only true when n is odd.

4.5.12

For 2×2 matrices, $det(cA) = c^2 det(A)$ not $cdet(A)$, so $det(A^{-1}) = \frac{1}{ad-bc} = \frac{1}{det(A)}$.

4.5.13 (first matrix)

$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ so $det(A) = 1$ as the product of the pivots.

4.5.14 (first matrix)

$$U = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \text{so } det(A) = 1 * 2 * 3 * 6 = 36.$$

4.5.15 (first matrix)

The first two steps of elimination reduces the first matrix to $\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, which clearly does not have linearly independent rows so its determinant is 0.

4.5.28

(a) For A, B square this is true as $det(A) = 0$ and $det(AB) = det(A)det(B) = 0$.

(b) This is false when a row exchange is used in reducing A to upper triangular form.

(c) False. $det(2I - I) = det(I) = 1$ but $det(2I) - det(I) = 2^2 - 1 = 3$.

(d) True. $det(AB) = det(A)det(B) = det(B)det(A) = det(BA)$.