HW9 Solutions

Math 2310

5.2.7

There are 5! = 120 permutation matrices and swapping the first two rows gives a 1 to 1 pairing of those with determinant +1 and those with determinant -1, so exactly 60 have determinant +1.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ requires 4 row exchanges to reach the identity.

5.2.9

By the permutation formula, any matrix of +1s and -1s has as its determinant a sum of 6 terms, each of which is +1 or -1. Any such matrix can be reached by starting with the matrix of all +1s and turning some of its entries negative. The determinant of the matrix with all +1s is 1+1+1-1-1-1 = 0, as in the sum there are 3 positive terms and 3 negative terms. Every time an entry in the matrix has its sign changed, one of the first three terms in this sum and one of the last three terms in this sum changes sign. That means either a +1 turns to a -1 and a -1 turns to a +1 in the determinant sum, two -1s turn into +1s, or two +1s turn into -1s. That means that from the original 3 +1s and 3 -1s in the determinant of the matrix with all +1s, changing a sign in one of the matrix entries moves that sum to 5 +1s and one -1, to one +1 and 5 -1s, or leaves it at 3 +1s and 3 -1s. These are the only reachable positions, as from 5 +1s and a single -1 there are no two -1s in the sum that can be flipped by changing a sign in the matrix (and similar for the opposite case). Therefore the only possible determinants of such a matrix are 1-5=-4, 3-3=0, and 5-1=4.

5.2.27

The cofactor formula gives $det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots$, where C_{11} does not depend on a_{11}, so the derivative of this expression as a function of a_{11} is the constant C_{11}.

5.2.34

(a) The first two and last three rows are the same.

(b) This question amounts to showing that every possible permutation of positions in the matrix includes at least one zero. Any permutation contains a 1 in exactly one position in column 5, exactly one position in column 4, exactly one position in row 1, and exactly one position in row 2. This accounts for 2 to 4 of the 1s in the permutation matrix (depending on if there's overlap between those specified by row and by column). Regardless, there is then at least one more 1 that is *not* in row 1, row 2, column 4, or column 5, and the corresponding position in A is therefore zero.

5.3.16

(a) $det \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = 10.$

(b-c) Each of these triangles is half of the parallelogram in (a), so they both have area 5.

5.3.19

The corresponding matrices are transposes so they have the same determinant, which is 4.

5.3.31

We want to compute $(u \times v) \cdot w$. $u \times v = (0, 0, 10)$ and $(0, 0, 10) \cdot (1, 2, 2) = 20$.

5.3.32

$$det\begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 12 + 0 + 0 \text{ - } 0 \text{ - } (\text{-}8) \text{ - } 0 = 20.$$

6.1.19

(a) The rank of B is the number of nonzero eigenvalues, 2.

(b) The determinants of both B and B^T are 0*1*2=0, so their product also has determinant 0.

(c) Eigenvalues are not able to be determined for general products of matrices.

(d) B^2 has eigenvalues $0^2, 1^2, 2^2$, so $0, 1, 4$. $B^2 + I$ then has eigenvalues $1, 2, 5$, and finally its inverse has eigenvalues $1, \frac{1}{2}, \frac{1}{5}$.

6.1.33

$Au = uv^T u = u(v^T u)$, where $\lambda_1 = v^T u$. Since the determinant of uv^T is 0, the other eigenvalue must be 0, so the trace is $\lambda_1 + \lambda_2 = v^T u + 0 = u_1 v_1 + u_2 v_2$ as expected.

6.1.34

the eigenvalues solve $0 = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda + 1)(\lambda - 1)(\lambda + i)(\lambda - i)$, so the eigenvalues are $1, i, -1, -i$. The eigenvectors for 1 are multiples of $(1, 1, 1, 1)$, the eigenvectors for -1 are multiples of $(1, -1, 1, -1)$, the eigenvectors for i are multiples of $(1, -i, -1, i)$, and the eigenvectors for $-i$ are multiples of $(1, i, -1, -i)$.

6.1.35

P always has determinant ± 1. The pivots are always 1 (after row exchanges). The diagonal has exactly three, one, or zero 1s, so the trace is always 3, 1, or 0. The eigenvalues of the identity are all 1, the eigenvalues of a permutation one row swap from the identity have $0 = (1 - \lambda)(\lambda^2 - 1) = -(\lambda - 1)(\lambda - 1)(\lambda + 1)$ so $\lambda = 1, 1, -1$, and the eigenvalues of a permutation two row swaps from the identity have $0 = -\lambda^3 + 1 = (1-\lambda)(\lambda^1 + \lambda + 1)$ so $\lambda = 1, \frac{-1\pm\sqrt{-3}}{2}$. Therefore the four numbers are $\pm 1, \frac{-1\pm\sqrt{-3}}{2}$.

6.2.4

(a) False, the eigenvalues can still have 0s.

(b) True

(c) True

(d) False, this tells us nothing about the eigenvectors of X.

6.2.14

The matrix A is not diagonalizable because the rank of $A - 3I$ is 1 not 0. Changing any entry except the 1 makes A diagonalizable.

6.2.26

Not every square matrix has n linearly independent eigenvectors because not all of the column space is necessarily spanned by eigenvectors, and also the nullspace and column space can intersect.

6.2.29

$AB = X\Lambda_1X^{-1}X\Lambda_2X^{-1} = X\Lambda_1\Lambda_2X^{-1} = X\Lambda_2\Lambda_1X^{-1} = X\Lambda_2X^{-1}X\Lambda_1X^{-1} = BA$, as multiplication of diagonal matrices is commutative.

6.2.31

$$(X\Lambda X^{-1} - \lambda_1 I)(X\Lambda X^{-1} - \lambda_2 I)\cdots(X\Lambda X^{-1} - \lambda_n I)$$

can be rewritten using $(X\Lambda X^{-1} - \lambda_i I) = X(\Lambda - \lambda_i I)X^{-1}$ to get

$$X(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)X^{-1}$$

Each of the parenthesized matrices is a diagonal with a zero in the appropriate entry: $\Lambda - \lambda_i I$ is 0 in the ith diagonal element, so their product is 0 in every entry, making the entire product 0.

6.2.37

If $B = A_2$, $BA_1A_2B^{-1} = A_2A_1A_2A_2^{-1} = A_2A_1$ ($B = A_1^{-1}$ would also work).

6.2.38

B is the eigenvector matrix and A must have n independent eigenvectors.