MATH 2310 Linear Algebra with Applications
Fall 2019 Exam #1

INSTRUCTIONS

- You have 75 minutes. If you finish within the last 15 minutes of class, please remain seated until the end so as not to disturb your classmates.
- The exam is closed book, closed notes, no calculators/computers/etc.
- Mark your answers ON THE EXAM ITSELF. If you are not sure of your answer, you may wish to provide a *brief* explanation so that we can at least know what you are trying to do. All short answer questions can be successfully answered in a few sentences at most. For full credit, be sure to show your work and justify your steps. Little credit will be given for correct answers without justification.
- Questions are not given in order of difficulty. Make sure to look ahead if stuck on a particular question.

Last Name	
First Name	
Cornell NetID (e.g. bwh59)	
All the work on this exam is my own. **(please sign)**	

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Q. 1	Q. 2	Q. 3	Q. 4	Q.5	Total
/20	/20	/20	/ 20	/20	/100

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1. (20 points) True/False

If true, give a short justification for why. If false, produce a counterexample and show why it contradicts the statement.

(a) If A is invertible, so is its transpose A^T.

(b) If $A = A^T$, then A is invertible.

(c) Given any vector $\vec{w} \in \mathbf{R}^3$, the set of all vectors $\vec{v} \in \mathbf{R}^3$ perpendicular to $\vec{w}$ (i.e. $\vec{w} \cdot \vec{v} = 0$) forms a subspace.

(d) Given any system of equations $A\vec{x} = \vec{b}$, the set of solutions $\{\vec{x} \mid A\vec{x} = b\}$ to the system, *if nonempty*, is a subspace.

(e) Let A be any matrix. If its nullspace $N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$ is equal to $\{\vec{0}\}$, then there must exist exactly one solution to $A\vec{x} = \vec{b}$.

Solution. (a) True. If A is invertible it is square with linearly independent rows and columns. The transpose of A is then also square and its rows are the columns of A which are linearly independent, so A^T is invertible. Another explanation is that if there exists A^{-1} with $AA^{-1} = I$ and $A^{-1}A = I$, then applying the transpose to each equation gives that $(A^{-1})^T A^T = I^T = I$ and $A^T(A^{-1})^T = I^T = I$, so $(A^{-1})^T$ is an inverse of A^T.

(b) False. A counterexample is the square matrix with 0 in every entry.

(c) True. If $\vec{w} \cdot \vec{v}_1 = 0$ and $\vec{w} \cdot \vec{v}_2 = 0$ then $\vec{w} \cdot (c\vec{v}_1 + d\vec{v}_2) = c(\vec{w} \cdot \vec{v}_1) + d(\vec{w} \cdot \vec{v}_2) = 0$, so the vectors perpendicular to $\vec{W}$ are closed under addition and scaling. A more geometric justification is that the vectors perpendicular to $\vec{w}$ in $\mathbf{R}^3$ form a plane containing the origin, which is always a subspace.

(d) False. If $\vec{b} \neq 0$ then $\vec{x} = 0$ is not a solution to $A\vec{x} = \vec{b}$, so those solutions cannot form a subspace.

(e) False. While it is the case that solutions to $A\vec{x} = \vec{b}$ are all of the form $\vec{x}_p + \vec{x}_n$ for $\vec{x}_p$ any particular solution and $\vec{x}_n$ in $N(A)$, there needs to be *some* solution $\vec{x}_p$ to the equation in the first place. For instance, the matrix A in the equation below has $N(A) = \{\vec{0}\}$ but the equation has no solutions:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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2. (20 points) Solving Systems of Linear Equations

(*Tip*: These are long-ish calculations; be smart about how you proceed and be neat and organized with your work. You almost certainly want to start on scratch paper.)

(a) (10 points) Find all solutions $\vec{x} = (x_1, x_2, x_3, x_4)$ to the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 4x_2 + 2x_3 + x_4 &= 0 \\ 4x_1 + 6x_2 + 3x_3 + 2x_4 &= 0 \end{aligned}$$

Solution. (a) First we reduce the matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 1 \\ 4 & 6 & 3 & 2 \end{bmatrix} \xrightarrow{R2-3R1}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -1 & -2 \\ 4 & 6 & 3 & 2 \end{bmatrix} \xrightarrow{R3-4R1}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -1 & -2 \\ 0 & -2 & -1 & -2 \end{bmatrix} \xrightarrow{R3-R2}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1+R2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we only need to solve $x_1 - x_4 = 0$ and $x_2 + \frac{1}{2}x_3 + x_4 = 0$. These give us $x_1 = x_4$ and $x_2 = -\frac{1}{2}x_3 - x_4$, so solutions are of the form

$$\begin{bmatrix} x_4 \\ -\frac{1}{2}x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \text{with } x_3 \text{ and } x_4 \text{ free variables.}$$

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(b) (10 points) Find all solutions $\vec{x}$ to the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ 3x_1 + 4x_2 + 2x_3 + x_4 &= 2 \\ 4x_1 + 6x_2 + 3x_3 + 2x_4 &= 3 \end{aligned}$$

Solution. First, looking at the equations we see that the third is the sum of the first two on both the left side and the right side, so the equations have a solution (this is the same as using elimination on the augmented matrix and checking that when the bottom row of A goes to 0 so does the bottom entry in $\vec{b}$). This could be solved the same way as part (a) by continuing with row operations, but a shortcut is to use the fact that solutions to $A\vec{x} = \vec{b}$ are all of the form $\vec{x}_p + \vec{x}_n$ for any particular solution $\vec{x}_p$ to the equation and $\vec{x}_n$ in the nullspace $N(A)$ that we found in part (a).

There are many possible particular solutions, but two simple ones are $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$.

So, using the first of those for $\vec{x}_p$, solutions are of the form $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. □

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3. (20 points) LU Factorization

We want to find the "LU factorization" of the following 2×4 matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 1 \end{bmatrix}$$

(a) (5 points) How many rows and columns should the "lower triangular" matrix L have?

(b) (5 points) How many rows and columns should the "upper triangular" matrix U have?

(c) (10 points) Find the LU factorization of A.

Solutions. (a) 2×2

(b) 2×4

(c) You only need to do one elimination step to get into echelon form, corresponding to

$$E_{21}(-2) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$

so $E_{21}(-2)A = U$, so

$$L = E_{21}^{-1}(-2) = E_{21}(2) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

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4. (20 points) The Number of Solutions to $A\vec{x} = \vec{b}$.

Given a matrix A of the following type, what are the possible numbers of solutions $\vec{x}$ to equations of the form $A\vec{x} = \vec{b}$? Be sure to justify your answers (e.g. with specific examples).

For example, if A is a 1×1 matrix, so $ax = b$ for some scalars $a, b \in \mathbf{R}$, we can have 0 solutions (e.g. $0 \cdot x = 1$), 1 solution (e.g. $2x = 3$), or infinitely many solutions (e.g. $0 \cdot x = 0$).

(a) (5 points) A a 3×3 matrix.

(b) (5 points) A a 2×4 matrix.

(c) (5 points) A a 4×3 matrix.

(d) (5 points) A a 4×4 matrix with at least one row of all zeros.

Solutions. These can be done from first principles, but the quickest way to get answers is via ranks.

(a) 0 ($b \notin C(A)$), 1 ($b \in C(A)$ and A invertible), ∞ ($b \in C(A)$, rank $A < 3$).

(b) Rank at most 2, so has to have nonzero nullity. Thus, 0 ($b \notin C(A)$) or ∞ ($b \in C(A)$).

(c) Rank at most 3, so can have nonzero nullity or not. Thus, 0 ($b \notin C(A)$) or 1 ($b \in C(A)$, rank 3) or ∞ ($b \in C(A)$, rank < 3).

(d) Rank at most 3, so nonzero nullity. So either 0 ($b \notin C(A)$) or ∞ ($b \in C(A)$).

There are of course many examples for each of these cases, but here are some simple ones.

(a) 1 solution: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∞ solutions: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

No solutions: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) ∞ solutions: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

No solutions: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

There are not enough equations for there to be a unique solution.

(c) 1 solution: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

∞ solutions: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

No solutions: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(d) ∞ solutions: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

No solutions: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Since the last equation must have 0 on the left hand side, there are not enough distinct equations for there to be a unique solution.

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5. (20 points) Finding Inverses. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 6 \\ 1 & 2 & 1 \end{bmatrix}.$$

(a) (10 points) Without explicitly calculating the inverse, show that A is invertible by using one of the criteria we have for invertibility.

(b) (10 points) Find the inverse A^{-1} of A. (This is a long computation, but relatively straightforward if you do it right, so be smart and organized with your work. You almost certainly want to try it on scratch paper first. Be sure to *check your answer* if you have time.)

Solution. (a) The easiest way is via elimination:

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 6 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 6 \\ 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -4 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -10 \end{bmatrix}$$

There are 3 nontrivial pivots, and so A is invertible.

(b)

$$A^{-1} = \begin{bmatrix} 9/10 & 1/5 & -21/10 \\ -2/5 & -1/5 & 8/5 \\ -1/10 & 1/5 & -1/10 \end{bmatrix}$$

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