MATH 2310 Linear Algebra with Applications
Fall 2019 Exam #2

INSTRUCTIONS

- You have 75 minutes. If you finish within the last 15 minutes of class, please remain seated until the end so as not to disturb your classmates.
- The exam is closed book, closed notes, no calculators/computers/etc.
- Mark your answers ON THE EXAM ITSELF. If you are not sure of your answer, you may wish to provide a *brief* explanation so that we can at least know what you are trying to do. All short answer questions can be successfully answered in a few sentences at most. For full credit, be sure to show your work and justify your steps. Little credit will be given for correct answers without justification.
- Questions are not given in order of difficulty. Make sure to look ahead if stuck on a particular question.

Last Name	
First Name	
Cornell NetID (e.g. bwh59)	
All the work on this exam is my own. **(please sign)**	

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Q. 1	Q. 2	Q. 3	Q. 4	Q.5	Total
/20	/20	/20	/ 20	/20	/100

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1. (20 points) Five True/False Questions on Eigenvalues

If true, give a short justification for why. If false, produce a counterexample and show why it contradicts the statement.

(a) If a square matrix A has 0 as an eigenvalue, then it is not invertible.

(b) If $AB = BC$ for some matrix B, then A and C have the same eigenvalues.

(c) If two $n \times n$ matrices A and C have the same eigenvalues, then they are similar.

(d) Let A be an $n \times n$ matrix whose entries are integers. All the eigenvalues of A must be real numbers.

(e) Let A be an $n \times n$ matrix that is diagonalizable whose eigenvalues are $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Then A^{100} has eigenvalues $\lambda_1^{100}, \lambda_2^{100}, \ldots, \lambda_n^{100}$.

Solution. (a) True. The determinant is a product of the eigenvalues, and so $\det(A) = 0$ and so is not invertible.

(b) False. Let B be a zero matrix and any A and C with different eigenvalues. (The key is that B has to be *invertible.*)

(c) False. An easy counterexample is the 3×3 identity matrix vs. a non-diagonalizable matrix with only 1 as an eigenvalue, which we saw in class.

(d) False. Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(e) True. We have $A = BDB^{-1}$ where D is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, so $A^{100} = BD^{100}B^{-1}$. Then use the fact that similar matrices have the same eigenvalues. □

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2. (20 points) Determinant

Calculate the determinant of

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & 1 & 2 & 0 \\ 3 & 1 & 2 & 5 \end{bmatrix}.$$

Solution. To find the determinant of A, there are several possible approaches: using minors along any row in A, the permutation formula, row reduction, even combinations thereof. I'll demonstrate how to use minors along the second row.

By the formula in the book, we have

$$det(A) = 0 * C_{21} + 2 * C_{22} + 0 * C_{23} + 3 * C_{24} = 2 * C_{22} + 3 * C_{24},$$

where $C_{22} = (-1)^{2+2} \det(M_{22}) = \det(M_{22})$ and $C_{24} = (-1)^{2+4} \det(M_{24}) = \det(M_{24})$.

$$M_{22} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

so $\det(M_{22}) = 10 + 0 + 8 - 0 - 15 - 24 = -21$.

$$M_{24} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

so $\det(M_{24}) = 2 + 0 + 3 - 2 - 0 - 9 = -6$.

So $\det(A) = 2 * (-21) + 3 * (-6) = -42 - 18 = -60$. $\square$

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3. (20 points) Orthonormal Basis and QR Factorization

Consider the vectors

$$\vec{v_1} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix}2\\1\\1\\7\end{bmatrix}$$

(a) Find an orthonormal basis for span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ via the Gram–Schmidt process.

(b) Produce a decomposition

$$\begin{bmatrix}\vec{v_1}\ \vec{v_2}\ \vec{v_3}\end{bmatrix} = QR$$

where Q is an orthogonal matrix (i.e. $Q^{-1} = Q^T$) and R is an upper-triangular matrix.

Solution. (a) I'll first find an orthogonal basis V_1, V_2, V_3 for $span\{v_1, v_2, v_3\}$ and then normalize to get q_1, q_2, q_3.

$$V_1 = v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}.$$

$$V_2 = v_2 - \frac{v_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

$$V_3 = v_3 - \frac{v_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{v_3 \cdot V_2}{V_2 \cdot V_2} V_2 = \begin{bmatrix}2\\1\\1\\7\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{7}{1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}.$$

Normalizing now gives $q_1 = \frac{1}{\sqrt{2}} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $q_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, and $q_3 = \frac{1}{\sqrt{3/2}}\begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}$.

(b) The QR factorization described in the book has $Q = [q_1\ q_2\ q_3]$ and

$$R = \begin{bmatrix} q_1 \cdot v_1 & q_1 \cdot v_2 & q_1 \cdot v_3 \\ 0 & q_2 \cdot v_2 & q_2 \cdot v_3 \\ 0 & 0 & q_3 \cdot v_3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 7 \\ 0 & 0 & 3/\sqrt{6} \end{bmatrix}$$

However, the question in part (b) asks for Q to be an *orthogonal* matrix, which means more than having orthonormal columns: it must be square and have orthonormal columns (you've shown on previous homework that this is the same as having $Q^{-1} = Q^T$). This means we need to find a fourth vector orthonormal to q_1, q_2, q_3, adjoin it to Q, and modify R to get the right dimensions and ensure that the product QR is still $[v_1\ v_2\ v_3]$.

To find q_4, recall that a vector orthogonal to the columns of a matrix lies in the nullspace of its transpose:

$$N([q_1\ q_2\ q_3]^T) = N([V_1\ 2V_3\ V_2]^T) = N\left(\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\right)$$

The first equality above is because $[q_1\ q_2\ q_3]^T$ and $[V_1\ 2V_3\ V_2]^T$ are related by row operations (scaling and a row swap) so they have the same nullspace (this just puts the matrix in an easier form to row reduce). The matrix then row reduces to:

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

The nullspace is then spanned by $(-1, 1, 1, 0)$ which normalizes to give $q_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

So now we have an orthogonal matrix $Q = [q_1\ q_2\ q_3\ q_4]$ and need to modify R so that $QR = [v_1\ v_2\ v_3]$. First of all, we want the product to be a 4×3 matrix, so as Q is 4×4, R should be 4×3. However, we already know from the usual QR factorization that the top 3 rows of R and the first 3 columns of Q multiply to give $[v_1\ v_2\ v_3]$, so we want R to just ignore the q_4 we added. This is done by making the 4th row of R all 0s, so

$$R = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 7 \\ 0 & 0 & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix}$$

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4. (20 points) Powers of a Matrix

Consider the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) (10 points) What are the eigenvalues and eigenvectors of A?

(b) (10 points) What is A^{100}?

Solutions. (a) The eigenvalues are $\lambda = 0, 1, 2$. (There are three distinct eigenvalues, so A is diagonalizable.) The 0-eigenspace is

$$\text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

The 1-eigenspace is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

The 2-eigenspace is

$$\text{span}\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

(b) We have

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 6 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and so

$$\begin{aligned} A^{100} &= \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 - 2^{101} & 1 - 2^{100} & 1 \\ -2 \cdot 3 + 3 \cdot 2^{100} & -2 + 3 \cdot 2^{100} & -2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

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5. (20 points) Orthogonal Projections.

Consider the plane in $\mathbf{R}^3$ given by

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) What is the matrix P that gives the orthogonal projection of a vector $\vec{v} \in \mathbf{R}^3$ onto U? Check that $P^2 = P$ and $P = P^T$.

(b) What is the closest point on U to $\vec{v} = \begin{bmatrix} 11 \\ 7 \\ 19 \end{bmatrix}$? Justify why your answer is the closest point.

Proof. (a) Set

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The projection matrix is

$$P = A(A^T A)^{-1} A^T = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

(Note that $P^2 = P$ and $P = P^T$.)

(b) We have

$$P \begin{bmatrix} 11 \\ 7 \\ 19 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 48 \\ 6 \\ 42 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 14 \end{bmatrix}.$$

There are many ways to justify why this is the closest point, but one is that the error $\vec{e} = \vec{v} - P\vec{v}$ is perpendicular to the plane we're projecting onto, so any point other that $\mathbf{P}\vec{v}$ on U will have to have a longer distance from v. ☐

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