NUMBERS

We begin with results about the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. On this sheet, "number" means integer. Some statements refer to the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ or the positive integers $\mathbb{Z}^+ = \{1, 2, \ldots\}$.

Definition 1. For two integers d, n we says that d **divides** n if there is an integer k such that $d \cdot k = n$. Here, d is called the divisor, n is the dividend, and k is the quotient. (Alternatively, we say that d is a factor of n, or n is a multiple of d.)

We write $d \mid n$ if d is a divisor of n or $d \nmid n$ if it is not.

Definition 2. An integer is **even** if it is divisible by 2, otherwise, it is is **odd**.

Remark 1. The notation $d \mid n$ signifies a relationship between the two numbers. It is different from the fraction d/n, which is a rational number. A question like "Does 2 divide 5?" makes sense, but "Does 2/5?" does not.

Exercise 3. (Parity and sign) Prove or disprove.

(1) If a number is even, then its negative is even. If a number is odd, then its negative is odd.
(2) If $d \mid a$, then $-d \mid a$ and $d \mid -a$. In addition, $d \mid |a|$. (Recall that the absolute value $|a|$ of a is a if $a \geq 0$ and $-a$ if $a < 0$.)

Exercise 4. (Parity and addition) Prove or disprove.

(1) The sum of two evens is even. The difference of two evens is even.
(2) The sum of two odds is odd. The difference of two odds is odd.
(3) Where $a, b \in \mathbb{Z}$, the number $a + b$ is even if and only if $a - b$ is even.
(4) Generalize the first item to arbitrary divisors.

Exercise 5. (Parity and multiplication) Prove the first, and prove or disprove the second.

(1) The product of two evens is even. Generalize to any divisor.
(2) The quotient of two evens, if it is an integer, is even.

Exercise 6. (Divisibility properties) Let d, m, and n be integers. Prove each statement.

(1) (Reflexivity) Every number divides itself.
(2) Every number divides zero. The only number that 0 divides is itself.
(3) (Transitivity) If $d \mid n$ and $n \mid m$, then $d \mid m$. That is, if n divides m, then so do n's divisors.
(4) (Cancellation) For $d, n \in \mathbb{Z}$, if for some nonzero integer a we have $ad \mid an$, then $d \mid n$. Conversely, if $d \mid n$, then $ad \mid an$ for all $a \in \mathbb{Z}$.
(5) (Comparison) For $d, n \in \mathbb{Z}^+$, if n is a multiple of d, then $n \geq d$.
(6) Every number is divisible by 1. The only numbers that divide 1 are 1 and -1.
(7) The largest divisor of a is $|a|$, for $a \in \mathbb{Z}$ with $a \neq 0$.
(8) Every nonzero integer has only finitely many divisors.

Exercise 7. What conclusion can you make if $a \mid b$ and $b \mid a$?

Exercise 8. Suppose that $a, b, c \in \mathbb{Z}$.

(1) Prove that if $a \mid b$, then $a \mid bc$ for all integers c.
(2) Prove that if $a \mid b$ and $a \mid c$, then a divides the sum $b + c$ and the difference $b - c$.
(3) (Linearity) Prove that if $a \mid b$ and $a \mid c$, then a divides any $i \cdot b + j \cdot c$, where $i, j \in \mathbb{Z}$.