LOGIC

Now that we've at least seen some proofs, let's get a better idea behind the structures that underly them. This is sometimes called **propositional logic**.

Definition 1. A (mathematical) **statement** (a.k.a. **proposition**) is a sentence that is either true or false.

For example, the sentence "all cars are green" is a (false) proposition. However, the sentence "$x = 1$" is *not* a proposition by itself, because we do not know what x is.

Given two propositions, we can form more complicated propositions using logical connectives.

Definition 2. Let A and B be propositions.

- (Negation) The proposition $\neg A$ ("not A") is true if (and only if)[1] A is false.
- (Conjunction) The proposition $A \wedge B$ ("A and B") is true if both A and B are true.
- (Disjunction) The proposition $A \vee B$ ("A or B") is true if at least one of A or B is true.
- (Implication) The proposition $A \Rightarrow B$ ("If A, then B" or "A only if B") is true if both A and B are true, or A is false.

Exercise 3. Can you express $A \Rightarrow B$ by using A, B, and the other logical connectives?

Exercise 4. Describe the meaning of $\neg(A \wedge B)$ and $\neg(A \vee B)$.

Exercise 5. What formula using A, B, and the logical connectives above, models an *exclusive* or ("xor"), which is what we colloquially mean by "or"?

Definition 6. A **truth table** is a table that illustrates all possible truth values for a proposition.

Exercise 7. Is $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$ always true?

Exercise 8. Are the following arguments logically correct?

(1) "If I'm guilty, I must be punished. I'm guilty. Thus I must be punished."
(2) "If I'm guilty, I must be punished. I'm not guilty. Thus, I must not be punished."
(3) "If I'm guilty, I must be punished. I must not be punished. Thus, I'm not guilty."
(4) "If I'm guilty, I must be punished. I must be punished. Thus, I'm guilty."

Definition 9. Two statements P and Q are (**logically**) **equivalent** (written $P \Leftrightarrow Q$ or "P if and only if Q") if P and Q have the same truth table.

Exercise 10. A coach promises, "If we win tonight, then I will buy you pizza tomorrow." Determine the case(s) in which the players can rightly claim to have been lied to.

Theorem 11 (DeMorgan's Law)**.** $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

Exercise 12 (DeMorgan's Law, II)**.** What is a similar statement for $\neg(A \vee B)$?

Definition 13. The **converse** of $A \Rightarrow B$ is $B \Rightarrow A$.

Exercise 14. Give an example of a true conditional proposition whose converse is false.

Definition 15. The **contrapositive** of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$.

Exercise 16. Find the converse and the contrapositive of the statement: "If a person lives in Ithaca, then that person lives in New York."

Theorem 17. *The implication $A \Rightarrow B$ is equivalent to its contrapositive.*

---

[1] All definitions are secretly "if and only if" statements, but it is standard to just write "if." Contesting this can lead into some murky philosophical waters.