INDUCTION

Axioms: (Peano's Postulates) The natural numbers are defined as a set $\mathbb{N}$ together with a unary "successor" function $S : \mathbb{N} \to \mathbb{N}$ and a special element $1 \in \mathbb{N}$ satisfying the following postulates:

(1) $1 \in \mathbb{N}$.
(2) If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.
(3) There is no $n \in \mathbb{N}$ such that $S(n) = 1$.
(4) If $n, m \in \mathbb{N}$ and $S(n) = S(m)$, then $n = m$.
(5) If $A \subset \mathbb{N}$ is a subset satisfying the two properties: (a) $1 \in A$ and (b) if $n \in A$, then $S(n) \in A$.

Then $A = \mathbb{N}$.

Theorem 1. *(Principle of Mathematical Induction) For each $n \in \mathbb{N}$, let $P(n)$ be a proposition. Suppose the follow two results hold:*

(a) *("Base case") The statement $P(1)$ is true.*
(b) *("Inductive step") If $P(n)$ ("the induction hypothesis") is true, then $P(S(n))$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

We'll first practice applying the theorem before attempting to prove it.

Exercise 2. Let n be a positive integer.

(1) We have $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
(2) We have $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$.
(3) We have $1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$.

Exercise 3. Given an integer $n \geq 4$, we have $n^2 \leq 2^n$.

Exercise 4. (Bernoulli's Inequality) If x is a real number such that $1+x > 0$, then $(1+x)^n \geq 1+nx$ for any $n \in \mathbb{N}$.

Exercise 5. Prove each by induction:

(1) For all $n \in \mathbb{N}$, the number $n^2 + n$ is even.
(2) For all $n \in \mathbb{N}$, the number $4^n - 1$ is divisible by 3.
(3) For all $n \geq 2$, the number $n^3 - n$ is divisible by 6. (Hint: Base 2.)
(4) If $n \in \mathbb{Z}_+$, then $(1 + \frac{1}{1})(1 + \frac{1}{2}) \cdots (1 + \frac{1}{n}) = n + 1$.

Exercise 6. Prove the Principle of Mathematical Induction (using Peano's postulates).

Exercise 7. A special chessboard is 2 squares wide and n squares long. Using n dominos that are 1 square by 2 squares, there are many ways to cover this chessboard with no overlap. How many are there? Prove your answer.

Exercise 8. Another chessboard is 2^n squares wide and 2^n squares long. Suppose that one of the squares has been cut out, but you don't know which one! You have a bunch of L-shaped pieces made up of 3 squares. Prove that you can cover this chessboard with L-shapes with no overlaps for any $n \in \mathbb{N}$.

Exercise 9. Show that the principle of mathematical induction implies the **well-ordering principle**, which says that any nonempty subset of the natural numbers has a smallest element. (Hint: Show by induction that if a set of natural numbers does not have a smallest element, then it is empty.)