SETS

Definition 1. [1] A **set** is an object S with the property that, given some other object x, precisely one of the statements: $x \in S$ ("x is in S" or "x is an element of S") or $x \notin S$ ("x is not in S") holds.

There are a couple of standard ways to write sets. For example, the set $A = \{1, 2, 3, 4\}$ contains precisely the four smallest positive integers, the set $B = \{3, 6, 9, \ldots, 3n, \ldots\}$ is the set of all positive multiples of 3, and the set $C = \{x \in \mathbb{N} \mid x \text{ is prime}\}$ is the set of prime numbers.

Definition 2. Let A and B be two sets.
We say that $A \subset B$ ("A is a subset of B") if $x \in A$ implies that $x \in B$.
We say that $A = B$ ("A equals B") if they contain precisely the same elements.

Exercise 3. We have $A = B$ if and only if $A \subset B$ and $B \subset A$.

Exercise 4. Let $A = \{1, \{2\}\}$. Is $1 \in A$? Is $2 \in A$? Is $\{1\} \subset A$? Is $\{2\} \subset A$? Is $1 \subset A$? Is $\{1\} \in A$? Is $\{2\} \in A$? Is $\{\{2\}\} \subset A$? Explain.

Definition 5. Let A and B be two sets. The **union** of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

and the intersection of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Exercise 6. For any two sets A and B, we have $A \subset A \cup B$ and $A \cap B \subset A$.

Definition 7. The **empty set** $\emptyset = \{\}$ is the set with no elements, that is, for any x, we have $x \notin \emptyset$.
We say that two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Exercise 8. If A is any set, then $\emptyset \subset A$.

Definition 9. Let A and B be two sets. The **complement of** B **relative to** A is the set

$$A \backslash B = \{x \in A \mid x \notin B\}.$$

When the set A is clear from context, this is sometimes just denoted B^c.

Exercise 10. (DeMorgan's law) Let X be a set and let $A, B \subset X$. Then
(1) $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$.
(2) $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.

Definition 11. Let A be B be two nonempty sets. The **Cartesian product** of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Exercise 12. If A contains n elements and B contains m elements, how many elements does $A \times B$ contain?

Definition 13. Let A be a set. The **power set** $\mathcal{P}(A)$ **of** A is the set of all subsets of A. In other words,

$$\mathcal{P}(A) = \{B \mid B \subset A\}.$$

Exercise 14. Let $A = \{1, 2, 3\}$. Identify $\mathcal{P}(A)$ by explicitly listing its elements.

[1] There are a lot of logical and philosophical implications behind the theory of sets, which are very interesting and an area of active research.