FUNCTIONS AND CARDINALITY

Definition 1. Let A and B be two nonempty sets. A **function f from A to B** (written as $f : A \to B$) is a subset $f \subset A \times B$ such that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$ (this condition is written as $f(a) = b$).

Exercise 2. Describe the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n$ as a subset of $\mathbb{Z} \times \mathbb{Z}$.

Definition 3. Let $f : A \to B$ be a function. The **domain of f** is A. If $X \subset A$, the **image of X under f** is the set

$$f(X) := \{b \in B \mid f(x) = b \text{ for some } x \in X\}.$$

If $Y \subset B$, then the **preimage of Y under f** is the set

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}.$$

Definition 4. (Basic Properties) A function $f : A \to B$ is:

- **surjective** (a.k.a. **onto**) if for every $b \in B$, there exist some $a \in A$ such that $f(a) = b$.
- **injective** (a.k.a. **one-to-one**) if for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- **bijective** (a.k.a. a **one-to-one correspondence**) if it is surjective and injective.

Exercise 5. Are the following functions injective? Surjective?

- Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(n) = n^2$.
- Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(n) = n + 2$.
- Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x) = x^2$.
- Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x) = x + 2$.

Exercise 6. If $f : A \to B$ is bijective, then there exists a bijection $g : B \to A$.

If two sets are in bijection with each other, we usually consider them to be identical as sets.

Notation 7. For any $n \in \mathbb{N}$, we define $[n] = \{1, 2, \ldots, n\} \subset \mathbb{N}$. Furthermore, we set $[0] = \emptyset$.

Definition 8. We say that a set A is **finite** if $A = \emptyset$ or if there exists an $n \in \mathbb{N}$ and a bijective correspondence between A and $[n]$. If this is the case, we say that the **cardinality of A** is n, and write this as $|A| = n$. If A is not finite, we say that A is **infinite**. If there is a bijection between two sets A and B, we say that A and B **have the same cardinality** (write this as $|A| = |B|$).

Exercise 9. Let A, B, and C be sets and suppose that there is a bijection between A and B, and a bijection between B and C. Then there is a bijection between A and C.

The following two results show that the cardinality of a finite set is well-defined.

Theorem 10. *(Pigeonhole Principle) Let $n, m \in \mathbb{N}$ such that $n < m$. There does not exist an injective function $f : [m] \to [n]$.*

Theorem 11. *Let A be a finite set and suppose that $|A| = m$ and $|A| = n$. Then $m = n$.*

Exercise 12. Let A and B be finite sets. If $A \subset B$, then $|A| \leq |B|$.

Exercise 13. (Inclusion/Exclusion Principle) Let A and B be two finite sets. Then

$$|A \cup B| + |A \cap B| = |A| + |B|.$$

Exercise 14. If A and B are two finite sets, then $|A \times B| = |A| \cdot |B|$.

Exercise 15. How many subsets does the empty set have? How many subsets does $[n]$ have?

Exercise 16. Show that if A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.