INFINITIES

Proposition 1. *The set $\mathbb{N}$ of natural numbers is* ***infinite*** *(i.e. not finite). (Hint: assume otherwise, so exists a bijection $f : [n] \to \mathbb{N}$ for some $n \in \mathbb{N}$. What can you say about the number $m := max(f(1), f(2), ...f(n)) + 1$?)*

Proposition 2. *Let A be an infinite set. If B is a set with the same cardinality as A, then B is infinite.*

What we'll see later is that surprisingly, the converse of the theorem does not always hold.

Theorem 3. *Let A be a set. The following statements are equivalent:*

(i) *The set A is infinite.*
(ii) *There exists an injective function $f : \mathbb{N} \to A$.*
(iii) *There exists a one-to-one correspondence between A and a proper subset of A.*

(There are many ways to show that the three statements are logically equivalent, e.g. proving $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$, or $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, etc.)

Definition 4. If A has the same cardinality as the natural numbers $\mathbb{N}$, we say that A is **countably infinite** or **cardinality** $\aleph_0$ ("aleph naught"). A set A is said to be **countable** if A is finite or countably infinite. A set is **uncountable** if it is not countable.

Example 5. The following sets are countable: the set of all odd numbers, prime numbers, the integers, all students in this class, all possible letter combinations using any finite alphabet.

Proposition 6. *If A is countable and $f : A \to B$ is a bijection, then B is countable.*

Proposition 7. *Every subset of a countable set is countable.*

Theorem 8. *The set of rational numbers is countable. (Hint: Make a table with column headings $0, 1, -1, 2, -2, \ldots$ and row headings $1, 2, 3, 4, \ldots$. Set the entry on the table at column m and row n to be the fraction m/n. Find a way to zig-zag through the table to hit every entry in the table (not the headings!) exactly once. This justifies that there is a bijection between $\mathbb{N}$ and the entries in the table. [Why?] Then appeal to the previous proposition.)*

Proposition 9. *If A and B are countable sets, then $A \cup B$ is countable.*

Theorem 10. *The open interval $(0, 1) \subset \mathbb{R}$ is not countable. Suppose for the sake of contradiction that there exists a bijection $f : \mathbb{N} \to (0, 1)$. For each $n \in \mathbb{N}$ its image under f is some number in $(0, 1)$. Let $f(n) := 0.a_{1n}a_{2n}a_{3n}\ldots$ where a_{1n} is the first digit in decimal form, a_{2n} is the second digit, etc. If $f(n)$ terminates after k digits, then our convention will be to continue the decimal form with 0's. Now, define $b = 0.b_1b_2b_3\ldots$ where*

$$b_i = \begin{cases} 2, & \text{if } a_{ii} \neq 2 \\ 3, & \text{if } a_{ii} = 2. \end{cases}$$

(a) *Prove that the decimal expansion that defines b is in* ***standard form****, where there is no k such that for all $i > k$, we have $b_i = 9$.*
(b) *Prove that for all $n \in \mathbb{N}$, we have $f(n) \neq b$. Explain why we have a contradiction.*

Question 11. *Let S be the set of infinite sequences of 0's and 1's. Is S countable or uncountable?*