GRAPHS

Definition 1. An (undirected) **graph** is a relation E on a set V that is symmetric and not reflexive. The elements of V are called **vertices**, and the elements of E are called **edges**. We often write the graph as $G = (V, E)$.

Many graphs can be drawn in the plane where vertices correspond to points and we draw a line segment between vertices when an edge exists.

Definition 2. Two vertices $v, w \in V$ are said to be **adjacent** if $(v, w) \in E$. We say that an edge $(v_1, v_2) \in E$ is **incident** to its endpoints $v_1, v_2 \in V$. Given a vertex $v \in V$, the number of edges incident to it is called its **degree** and is denoted by $\deg(v)$.

Proposition 3. *Let $G = (V, E)$ be a graph. Then $|E| = \frac{1}{2}\sum_{v\in V} \deg(v)$.*

Definition 4. Given $v, w \in V$, a **(v, w)-walk in G** is an alternating sequence of vertices and edges, starting with v and ending with w, such that any two consecutive vertices are distinct and each edge is incident to the two vertices it lies between in the sequence.

We say the graph G is **connected** if there is a (v, w)-walk between any two distinct $v, w \in V$.

We say a (v, w)-walk is a **path** if all vertices on the walk are distinct. If $v = w$ but all other vertices are distinct, we say that the (v, w)-walk is a **cycle**.

A graph G is called a **forest** if it contains no cycles. A connected forest is called a **tree**.

Proposition 5. *If every vertex in a finite graph G has degree at least 2, then G contains a cycle.*

Theorem 6. *Let G be a finite tree. Then $|E| = |V| - 1$. (Hint: use induction and the previous proposition).*

Definition 7. A graph G is **planar** if it can be drawn in the plane such that no two edges intersect each other (except possibly at endpoints). The regions F in the plane that are created by a plane drawing are called **faces**.

Proposition 8. *Let $G = (V, E, F)$ be a connected planar graph. Assume that $|V| \geq 3$. Then $|E| \geq \frac{3}{2}|F|$.*

Theorem 9. *(Euler's formula) Let $G = (V, E, F)$ be a connected planar graph. Then*

$$|V| - |E| + |F| = 2.$$

(Hint: induct on faces; use the theorem above; for inductive step, consider an edge on a cycle)

Theorem 10. *Let $G = (V, E, F)$ be a planar graph and assume that $|V| \geq 3$. Then $|E| \leq 3|V| - 6$. (Hint: Assume WLOG that it's connected, use previous two results.)*

Theorem 11. *Let $G = (V, E, F)$ be a planar graph such that $|V| \geq 3$ and no face of G is a triangle (i.e. bounded by three edges). Then $|E| \leq 2|V| - 4$.*

Proposition 12. *Every planar graph has a vertex v such that $\deg(v) \leq 5$. (Hint: degree + Euler)*

Definition 13. The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G so that adjacent vertices have different colors.

Theorem 14. *Every planar graph has chromatic number at most 6. (Hint: Use Prop. 12)*

Remark 1. It turns out that every planar graph has chromatic number at most 4. This is one of the great triumphs of late 20th century mathematics.