AFFINE AND PROJECTIVE PLANES

Relations and graphs are some of the simplest ways to model relationships and are useful because they are so natural. However, some structures are much less intuitive, but are tremendously powerful once grasped. We'll practice exploring and exploiting the power of a more complicated axiomatic definition on this sheet.

One way to think about affine/projective planes is that they're kind of a "balanced system of relations" between "point objects" and "line objects." They are ubiquitous once you know how to look for them, with the applications running from photography (e.g. calibrating cameras, optimizing pictures) to the modern design of experiments and schedule optimization.

Definition 1. An **affine plane** $\Pi = (\mathcal{P}, \mathcal{L})$ is a set $\mathcal{P}$ of **points** and a set $\mathcal{L}$ of **lines** together with an **incidence relation** (which determines which points lie on which lines) that satisfy the following axioms: (Recall that two lines L_1 and L_2 are *parallel* if they do not intersect.)

- (A1) Every line contains at least two points.
- (A2) Every pair of points lie on a unique line.
- (A3) For any line L and any point p that is not on L, there exists a unique line M that goes through p and that is parallel to L;
- (A4) There are three non-collinear points (i.e. that don't lie on the same line).

Examples 2. These are all affine planes: (1) $\mathcal{P} = \mathbb{R}^2$ and $\mathcal{L}$ is the set of lines in $\mathbb{R}^2$; (2) more generally, any plane $\mathcal{P}$ and $\mathcal{L}$ the lines in that plane; (3) $\mathcal{P}$ the set of one-dimensional subspaces of $\mathbb{R}^3$ than do not lie on the xy-plane, and $\mathcal{L}$ the two-dimensional subspaces of $\mathbb{R}^3$ other than the xy-plane.

However, some of the most interesting affine planes have *finite* sets $\mathcal{P}$ and Ł.

Question 3. *What is the minimal number of points an affine plane can have? (Play around with the axioms. For example, (A4) tells you that you must have at least 3 points.)*

Definition 4. A **projective plane** $\Pi = (\mathcal{P}, \mathcal{L})$ is a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines together with an incidence relation such that

- (P1) Every line contains at least two points.
- (P2) Any two distinct points lie on a unique line;
- (P3) Any two distinct lines intersect at a unique point;
- (P4) There are four points, no three of which are collinear.

Example 5. An example of a projective plane: $\mathcal{P}$ the antipodal points on a sphere centered at the origin (i.e. equivalence classes of the sphere under the relation that $x \sim y$ if there is a line going through x, y, and the origin), and $\mathcal{L}$ great circles on the sphere (i.e. circles of the same length as the "equator," and not, say, those of different latitudes).

Definition 6. We say that an affine or projective plane $\Pi = (\mathcal{P}, \mathcal{L})$ is **finite** if $\mathcal{P}$ and $\mathcal{L}$ are both finite.

Proposition 7. *There are at least three points on any line in a projective plane.*

Proposition 8. *Let* $\Pi = (\mathcal{P}, \mathcal{L})$ *be a projective plane and fix a line* L *in* $\mathcal{L}$. *Now set* $\mathcal{P}' = \mathcal{P} - L$ *and let* $\mathcal{L}'$ *be all nonempty sets of the form* $M - L$ *where* M *is a line in* Π. *Then* $\Pi' = (\mathcal{P}', \mathcal{L}')$ *is an affine plane.*

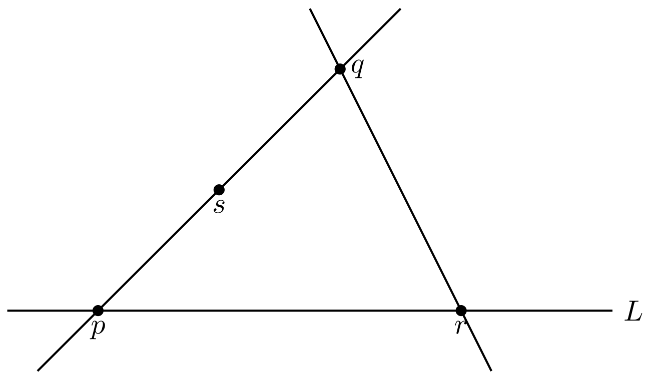
Question 9. (1) *Describe the the smallest affine plane, i.e. the affine plane with the smallest number of points and lines. (Starting from axiom (A3), explain how axioms (A1) and (A2) force you to add points and lines.)*

(2) *Describe the smallest projective plane using the same strategy.*

Question 10. *Let* Π *be a finite projective plane. We want to prove that the number of lines through any point is the same as the number of points on any line. We will walk through a proof of this.*

(1) *Let* L *be any line and* p *any point that is not on* L*. Construct a bijection between the set of lines through* p *and the set of points on* L*. Make sure you prove that this map is well-defined, injective, and surjective.*

(2) *Extend the result to the case where the point* p *is on the line* L*. You can use the picture below as a source of inspiration.*



(3) *Now prove, with little effort, that*
- *the same number of lines pass through every point of* Π*;*
- *the same number of points lie on every line of* Π*.*

The previous question inspires the following definition.

Definition 11. The **order** of Π is the number q such that
- there are $q + 1$ lines through every point of Π;
- there are $q + 1$ points on every line of Π.

Question 12. *Let* Π *be a finite projective plane of order* q*.*

(1) *Show that the number of points in* Π *is equal to the number of lines in* Π*. (Hint: Find two different ways to count pairs* (p, L) *of points and lines where* $p \in L$*.)*

(2) *How many points are in* Π*? (Hint: Fix a point, count the lines through that point, and count the number of points on each of these lines. Did you count all the points? Did you count any of them more than once?)*

Question 13. *Give an example of a structure that satisfies axioms (P1) and (P2), where there are two lines with a different number of points that lie on them.*