## WHERE WE'VE COME FROM AND WHAT LIES BEYOND

To wrap this little chapter (and our class) up, we'll explore where the surreal numbers came from, and how Conway came to discover them. Famously, they came from game-playing, and were seemingly "hidden in plain sight"!

There is an alternative way to think about surreal numbers. It turns out that every surreal number represents a two-player "game." Games turns out to be even more general than surreal numbers.

We can think of any surreal number $x = \{L|R\} = \{x^L|x^R\}$ as a two-player game where the left set represents moves that one player (Left) can make and the right set represents moves that the other player (Right) can make.

For example, if Left starts the game and moves to $x^L \in L$, then the game becomes represented by $x^L$. Then Right can move to any number (game state) $(x^L)^R$, in which case Left can move to any number in $((x^L)^R)^L$ and so on.

A game ends when a player cannot make a move. For example $\{|1, 2, 3\}$ represents a game here the left player cannot make a move. Under the *normal play convention*, the player that cannot make a move loses. We'll follow this convention, as most games (e.g. chess) follow it.[1]

We can model these games in many ways, but one way, also invented by Conway, is to use a game called *Hackenbush*. It's a two-player game played with a graph that has edges of two colors (say, blue and red) and is connected to a base line ("the ground"). Left and Right alternate turns, and Left can only delete blue edges, while Right can only edge red edges.

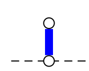
After one edge is deleted, any edge that is no longer connected to the ground is deleted. The last player who deletes an edge wins.

Let's see some examples.

**Example 1.** Consider the game

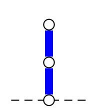
with no edges. It's called the *endgame*. Neither player can make a move, so the game is $\{|\} = 0$. The first player to move here loses.

**Example 2.** If there is just one blue edge,



then Left can move to 0, while Right has no moves, so the game is $\{0|\} = 1$.

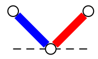
**Example 3.** If there are two blue edges stacked on top of one another,



then Left can pull the bottom edge to get game 0, or the top edge to get game 1, so this game is $\{0, 1|\} = 2$. If we had two red edges instead, we would have $\{|0, -1\} = -2$.

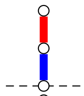
[1]The opposite convention—where the player that cannot make a move wins—is called the *misère* convention. The interesting and surprising thing about this is that despite the fact that it's just the opposite of normal, the strategies to win under the *misère* convention are usually drastically different from that under normal play.

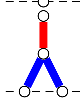
**Example 4.** What about the game



there is one red edge and one blue edge, so our game is $\{-1|1\} = 0$ because the first player to move will lose as the second player will be able to move to the endgame.

**Question 1.** *What are the values of the following games?*

(a) 

(b) 

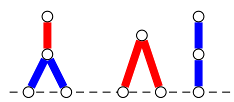
We say $G > 0$ ("$G$ is positive") if there is a winning strategy for Left.

We say $G < 0$ ("$G$ is negative") if there is a winning strategy for Right.

We say $G \equiv 0$ ("$G$ is like 0") if there is a winning strategy for the *second* person to move. We say $G||0$ ("$G$ is fuzzy") if there is a winning strategy for the first person to move.

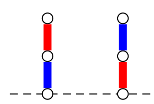
These have their natural consequences. For example, if $G \geq 0$, then if Right starts, then there is a winning strategy for Left (as Left would move second). Or $G <_{||} 0$ (which means $G < 0$ or $G||0$) which says that if Right starts, then there is a winning strategy for Right.

Now comes the question that led Conway to his theory. What happens if instead of just having one game, we are two players playing multiple instances simultaneously?



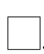
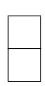
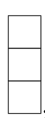
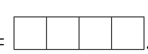
It turns out that the answer is to assign surreal numbers to each of these (say, three as above) separate games and then take their (surreal number) sum!

**Question 2.** *Use surreal numbers to show that the second player to move wins the following Hackenbush game:*

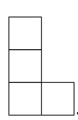


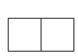
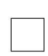
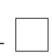
**Question 3.** *How do you represent the negative of a Hackenbush game? Suppose we play a Hackenbush game and its negative, so its sum is 0. What is a strategy that the second player can use to guarantee a win?*

This applies to many other kinds of two-player games. For example, we could consider a game of dominos, in which you're given a shape that you have place dominos, where Left has to place dominos vertically and Right has to place them horizonally where

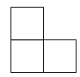
$0 =$ , $1 =$  $=$ , $-2 =$ , etc.

but what's interesting is determining a value of a region like

.

Left has one (dumb) move to $-1 =$  and another (better) move to  $+$  $= 0$, whereas Right has just one move to $1 =$ . Thus, the value of the game is $\{0, -1|1\} = \frac{1}{2}$, so this region is an advantage of exactly half a move for Left! So note that playing a game of this on any shape reduces to find the sums of possible sub-games. Try something like a general $n \times m$ rectangle or a large jagged region for a challenge!

Finally, as mentioned, above, games are more general than numbers. For example, using this domino game setup, we have configurations like



which corresponds to the game $\{0|0\}$. This is not a surreal number, but is a game. We call $\{0|0\} = *$ ("star") and it corresponds to an unconditional win for the first player, which you can easy tell from this form of the game; it is fuzzy with 0. If you insist on interpreting it as a number, $*$ is neither positive nor negative (it is its own negative), but it less than all positive rationals and greater than all negative rationals. (How does it compare with something like $\frac{1}{\omega}$?)

This procedure should look somewhat familiar: our games from the beginning of class! But those games were slightly different: in the games above, Left and Right had different moves, whereas for the games we first considered, both players had the *same* moves. This means that we should be looking at games that look more like $*$, where our sets $L$ and $R$ have the same moves, so let's write

$$*0 = \{|\} = 0, *1 = \{*0| * 0\} = \{0|0\} = *, *2 = \{*0, *1| * 0, *1\}, \ldots$$

and we'll abbreviate $*n =: \{*0, *1, \ldots, *(n-1)\}$ as both the left and right sets of these games are the same. Such games are called *nimbers*.

Nimbers are games that have their own arithmetic. We have the identity $*n + *n = 0$ for any $n$, by the proof Question 3 above. It turns out that $*n + *m$ is given by surreal number addition, but where the *smallest missing* $*k$ (for $k \in \mathbb{N}$) is its value. For example, $*0 + *n = *n$ and $*1 + *2 = \{*1 + *1, *1 + *0, *2 + *0\} = \{*0, *1, *2\} = *3$ but $*1 + *3 = \{*1, *0, *1 + *2, *3 + *0\} = \{*1, *0, *3, *3\} = *2$.

**Question 4.** *Remember our bridge standoff game from the very first class assignment*[2]*. What is the nimber associated with the original game of length 8? Of length 21? For arbitrary $n$?*

In this context, for a player to find a winning move, they must find a move which moves to position $*0 = \{|\}$. And of course, these would just say that a winning strategy *exists*; if you play suboptimally, you can still lose an actual game.

**Question 5.** *What about if we modify our game like in our first groupwork assignment*[3] *to jump up to $k$ steps instead of just 3? Let's write $B(k, n)$ for the bridge standoff game where you're allowed to move* 1 *to $k$ steps on a bridge of length $n$. What is the nimber corresponding to the game $B(k, n)$? What should be your strategy?*

Of course, you should check that the values of your games match up with the solutions that you obtained for these problems.

**Question 6.** *Say you are two opposing generals that have to play* 10 *games of bridge standoff game simultaneously. If all these games are the same $B(k, n)$, for which $k$ and $n$ should you accept this challenge? How should you act?*

*Suppose you had a choice that you could choose $k$ or $n$, while your opponent chooses the other. If you're picking these simultaneously (or via a process that is hidden until choices are set), which should you choose? What if you had to pick first and your opponent could respond? What if you could pick second?*

[2] `http://pi.math.cornell.edu/~bhwang/3040/bridge_standoff.pdf`
[3] `http://pi.math.cornell.edu/~bhwang/3040/hw2.pdf`

For a challenge, think about what you could do if you could also choose the number $m$ of games to play. Or suppose that if you could choose to pick one of $m$, $k$, or $n$. There are many variations that are fun to think about. For example, suppose that you are given the choice of two of them randomly, with the caveat that your opponent can pick the remaining one in response, should you take this challenge? Of course, you can also try and think of cases where the games $B(k, n)$ themselves vary. All of these are answerable with the surreal number formalism. Probably the most interesting question here is to try and consider yourself as an impartial arbiter of this challenge: is there some way you can define a game using successive bridge standoffs and choices of variables so that it is fair?

Of course, there is more to this story. As we can't apply "$\geq$" to $*$ and 0, if we make the surreal numbers bigger by allowing for $*$, we lose the fact that such elements have an ordering, so for certain puposes it's better to restrict to the surreals. (But of course, for games, that's not a problem as we know precisely what $*$ means for a game.) However, once we allow for $*$, we can get further games like $\uparrow = \{0|*\}$ and $\downarrow = \{*|0\}$ and remarkable identities like $\{0|\uparrow\} = \uparrow + \uparrow + *$, but this is probably a good place to conclude our introduction.

There's a whole wide world out there to explore, and hopefully with some of skills we've developed here, you will have a good foundation for you to proceed on your own journeys!