3040 HW 2

1. (a) Let $n, n+1, n+2$ be three arbitrary consecutive integers. Then their sum is $n + (n+1) + (n+2) = 3(n+1)$, which is divisible by 3 by definition.

   (b) Let $a, b, m \in \mathbb{Z}$ and assume $ab \mid m$. By definition, $m = (ab)k$ for some integer k. Thus, since $m = a(bk)$ and bk is also an integer, $a \mid m$ by definition. Similarly, noting ak is an integer shows that $b \mid m$.

   (c) We prove the contrapositive. Let x odd so $x = 2k+1$ for $k \in \mathbb{Z}$. Then $x^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $(2k^2 + 2k) \in \mathbb{Z}$, then x^2 is odd as well.

2. (a)

$$\begin{aligned}
(m-n) - (p-q) &\overset{1.25(i)}{=} (m-n) + ((-p) + -(-q)) \\
&\overset{1.22(i)}{=} (m-n) + ((-p) + q) \\
&\overset{1.1(i),(ii)}{=} (m+q) + ((-n) + (-p)) \\
&\overset{1.25(i)}{=} (m+q) - (n+p)
\end{aligned}$$

   (b)

$$\begin{aligned}
(m-n)(p-q) &\overset{1.11(i)}{=} (mp + (-n)p) + (m(-q) + (-n)(-q)) \\
&\overset{1.25(iii)}{=} (mp - np) + (-(mq) + (-n)(-q)) \\
&\overset{1.20}{=} (mp - np) + (-(mq) + nq) \\
&\overset{1.1(i),(ii)}{=} (mp + nq) + (-(mq) + -(np)) \\
&\overset{1.25(i)}{=} (mp + nq) - (mq + np)
\end{aligned}$$

   (c) By $1.25(i)$ we distribute the negative showing $(m-n) - (p-q) = (m-n) + ((-p) + -(-q))$. Since $q + (-q) = 0$, $-(-q) = q$, and thus $(m-n) + ((-p) + -(-q)) = (m-n) + ((-p) + q)$. By associativity and commutativity of addition, we rearrange $(m-n) + ((-p) + q) = (m+q) + ((-n) + (-p))$. Another application of $1.25(i)$ then shows $(m+q) + ((-n) + (-p)) = (m+q) - (n+p)$ as hoped.

3. Let P1 be the player going first and P2 the player going second.

   (a) Let n be the length of the bridge and 3 be the maximum step size. By (b) we have that P2 has a winning strategy if $4|n$, and $P1$ has a winning strategy otherwise.

   (b) Let n be the length of the bridge and k be the maximum step size.

   Claim 1: P2 has a winning strategy if $(k+1)|n$

   Proof: By assumption, $n = (k+1)p$ for some positive integer p. We induct on p:

   When $p = 1$, initial distance between P1 and P2 is $k+1$. P1 must first move $x \in \{1, \dots, k\}$ steps, reducing the distance to $(k+1-x) \in \{1, \dots k\}$. Thus P2 may move $k+1-x$ steps to win the game.

   Assume P2 has a winning strategy on a bridge of length $(k+1)p$. If the bridge has initial length $(k+1)(p+1) = (k+1)p + (k+1)$ then P1 must first move $x \in \{1, \dots k\}$ steps, reducing the distance to $(k+1)p + (k+1) - x$ steps. Let P2 then move $(k+1-x) \in \{1, \dots k\}$. As it is now P1s turn and the distance between them is $(k+1)p$, P2 has a winning strategy by induction.

   Claim 2 P1 has a winning strategy if $(k+1) \nmid n$

   Proof: If $(k+1) \nmid n$, then $n = p(k+1) + r$ for some $r \in \{1, \dots k\}$. So, let P1 first move r steps. The game is now equivalent to one of length $p(k+1)$, but with P2 going first. Thus from claim 1, P1 has a winning strategy.