MATH 3040: HOMEWORK 6

All answers must be in complete sentences. All assertions must be proven, per the student contract. Remember that this is a writing class, so while correctness and thoroughness are most important, *style* matters as well.

1. (Practice in Posing Questions) What is an interesting open-ended question you could ask, which is related to the material that we covered in class? What is a concrete mathematical question that could be used to approach an answer to the question?

2. (Images, and Prelim I wrap-up) Let $f : A \to B$ be a function. Let C and D be subsets of A.

(a) Prove that $f(C \cup D) = f(C) \cup f(D)$.
(b) Prove that $f(C \cap D) \subset f(C) \cap f(D)$.
(c) Give an explicit example of a function f in which $f(C \cap D)$ is a *proper* subset of $f(C) \cap f(D)$ (i.e. where (b) holds, but where $f(C \cap D) \neq f(C) \cap f(D)$).

3. (A cool bijection) Prove that the function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ defined by

$$f(m, n) = 2^{m-1}(2n - 1)$$

is a bijection.

4. (Properties about sets, functions, and cardinality) Prove each of the following statements.

(a) Let A and B be nonempty finite sets. The following statements are equivalent:
  (i) There is an injection from A to B.
  (ii) There is a surjection from B to A.
  (iii) $|A| \leq |B|$.
(b) Let A and B be nonempty sets such that $|A| = |B|$, and let $f : A \to B$ be a function. If f is injective or surjective, then f is a bijection.
(c) Let A and B be nonempty finite sets. Show that at least one of the following statements is always true: (1) There is an injection from A to B; (2) There is a surjection from A to B.

The following question is to be done as a group. (Think about this over the weekend; new groups will be assigned next week.)

5. (Groupwork) A familiar way to get functions $f : \mathbf{R} \to \mathbf{R}$ is to define $f(x)$ to be a polynomial, such as $f(x) = x^3$. These are more or less the subject of a first course in calculus, if we assume that f is continuous and differentiable (and all polynomials are both). We now want to study the properties of these polynomial functions in the context of a function between sets.

(a) Prove that every linear polynomial function $f : \mathbf{R} \to \mathbf{R}$ is a bijection. (A linear polynomial is a degree-one polynomial, i.e. $f(x) = ax + b$ for $a, b \in \mathbf{R}$.)

(b) Prove that no quadratic polynomial function $f : \mathbf{R} \to \mathbf{R}$ is a bijection. (A quadratic polynomial is a degree-two polynomial, i.e. $f(x) = ax^2 + bx + c$ for $a, b, c \in \mathbf{R}$.)

(c) (This question requires a little calculus.) Find a simple "if and only if" statement in terms of the coefficients $a, b, c, d \in \mathbf{R}$ that characterizes when a cubic polynomial function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = ax^3 + bx^2 + cx + d$$

is a bijection, and prove that your characterization is correct. (This question can be tricky. Play around with some examples and see if you can find any patterns.)