MATH 3040: HOMEWORK 7

All answers must be in complete sentences. All assertions must be proven, per the student contract. Remember that this is a writing class, so while correctness and thoroughness are most important, *style* matters as well.

1. (Practice in Posing Questions) What is an interesting open-ended question you could ask, which is related to the material that we covered in class? What is a concrete mathematical question that could be used to approach an answer to the question?

2. (Prove or disprove and salvage) Suppose that we have functions $f : A \to B$ and $g : B \to C$, and define $h = g \circ f : A \to C$. Prove or disprove and salvage each of the following statements.
   (a) If h is injective, then f is injective.
   (b) If h is injective, then g is injective.
   (c) If h is surjective, then f is surjective.
   (d) If h is surjective, then g is surjective.

3. (Cardinality)
   (a) Pick out 20 distinct numbers from the set
$$\{1, 4, 7, 10, 13, 16, 19, \ldots, 97, 100\}.$$
   Prove that no matter how you make those choices, you will have chosen two numbers whose sum is 104.
   (b) Create an injection from $\mathcal{P}(\mathbf{N})$ (the power set of the natural numbers) to the real numbers $\mathbf{R}$.

4. (Countability)
   (a) Every countable union (i.e. a union of countably many sets) of countable sets is countable. (Hint: there are three cases to consider finite unions of countable sets [see Prop. 9 on *Infinities*], countably infinite unions of finite sets, and countably infinite unions of countably infinite sets.)
   (b) If A and B are countable sets, then $A \times B$ is countable.
   (c) The set of all finite sequences of 0's and 1's (e.g. 011000110 is one such sequence) is countable.

The following question is to be done as a group.

6. (Groupwork) ("WLOG") (a) Consider the following game played by
two players A and B, where the players take turns and A goes first. You
place four points (labelled 1, 2, 3, 4) on a chalkboard such that no three
of the points lie on the same straight line. On each player's turn, they
must pick two of the four points that have not been connected and draw
a straight line segment between them. However, a player is not allowed to
form a triangle whose vertices are among the original four starting points.
(If you accidentally form a triangle by criss-crossing lines, but where the
vertices of that triangle are not all among the four starting points, that is
fine.) The first player unable to move loses.

Which player has a winning strategy? (Hint: To simplify things, you must
understand how certain positions are "essentially the same"; for example,
all moves by A are essentially the same, so one can assume that the move
connects points 1 and 2. Then B has essentially two distinct choices: draw
a segment adjacent to the A's segment or not.)

(b) What if you have the same setup, but where your goal is to *create* the
first triangle. Which player has a winning strategy?