MATH 3040: HOMEWORK 9

All answers must be in complete sentences. All assertions must be proven, per the student contract. Remember that this is a writing class, so while correctness and thoroughness are most important, *style* matters as well.

1. (Practice in Posing Questions) What is an interesting open-ended question you could ask, which is related to the material that we covered in class? What is a concrete mathematical question that could be used to approach an answer to the question?

2. (Actually About Graphs: Prove or Disprove and Salvage)

(a) In every group of five people, there are two people who know the same number of people in the group. (Assume throughout that "knowing someone" is mutual; if you know them, they they know you.)
(b) In every group of five people, at least one of the following groups exist: a group of three "friends" who all know each other, or a group of three "total strangers" in which no two members of this group know each other.
(c) In every group of six people, at least one of the following groups exist: a group of three "friends" or a group of three "total strangers."

3. (Questions about Groups)

(a) [TAP, Prop. D.13] The even permutations constitute a subgroup of S_n. Show that this subgroup A_n contains $n!/2$ members.
(b) [TAP, Prop. D.19] The multiplicative group of positive rational numbers is not finitely generated.
(c) [TAP, Prop. D.21] Show that every finitely generated group is countable.

4. (Questions about Graphs and Groups)

(a) [TAP, Prop. D.23] Show that if G is a finitely generated group with a fixed generating set S that does not contain 1, then the Cayley graph $\Gamma(G, S)$ is a connected graph.
(b) [TAP, Proj. D.27] Draw the Cayley graph for $\mathbf{Z}$ for the set of generators $\{1, 2\}$. (And, of course, justify why your answer is correct.)

5. (Groupwork, due May 8) A graph on n vertices is said to be **complete** if all pairs of distinct vertices are adjacent.

(a) If G is a complete graph of 5 vertices, how many cycles of length 3 are there? Length 4? Find and prove a formula for the number of cycles of length k (they are called **k-cycles**). Note that no graph has cycles of length 1 or 2, and that for sufficiently large k, there are no k-cycles on G.

(b) Find and prove a formula for the number of k-cycles on a complete graph G of n vertices.

(c) Suppose that G is obtained by taking a complete graph on n vertices and removing an edge. How many k-cycles are in G?