MATH 3040 Prove It!
Spring 2020

Preliminary Exam #1

INSTRUCTIONS

- You have 50 minutes.
- The exam is closed book, closed notes, no calculators. However, you are allowed a one-page (front and back) "cheat sheet." **If you use such a sheet, submit it with your exam.** You are free to apply any result that we covered in class or on the homeworks, unless the problem explicitly tells you to use a certain approach. You do not need to cite the name and number of such results, just be clear on which result you are using.
- Mark your answers ON THE EXAM ITSELF (in particular, no exam books or loose sheets of paper). If you are not sure of your answer, you may wish to provide a *brief* explanation so that we can at least know what you are trying to do. For full credit, be sure to justify your steps.
- Write your name on the top of each page with a problem listed.
- Questions are not given in order of difficulty. Make sure to look ahead if stuck on a particular question.

Last Name	
First Name	
Student ID	
All the work on this exam is my own. **(please sign)**	

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Q. 1	Q. 2	Q. 3	Q.4	Total
/10	/20	/20	/20	/70

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1. (10 points) Contrapositive and Negation

Consider the statement

"$\forall x \in \mathbf{R}, (x \geq 3) \Rightarrow (x^2 > 5)$"

The following statements do not require proof, but be absolutely sure of your answer, as no partial credit will be awarded. (Think about the truth tables, if that helps.)

(a) (5 points) Write the contrapositive of the statement.

(b) (5 points) Write the negation of the statement.

Solution. (a) $\forall x \in \mathbf{R}$, $(x^2 \leq 5) \Rightarrow (x < 3)$

(b) $\exists x \in \mathbf{R}$ such that $x \geq 3$ and $x^2 \leq 5$. □

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2. (20 points) Sets and Power Sets.

Given a set A, let $\mathcal{P}(A)$ denote its powerset.

(a) (10 points) Let S and T be sets. Show that $\mathcal{P}(S) \cup \mathcal{P}(T) \subseteq \mathcal{P}(S \cup T)$.

(b) (5 points) Show that it is not always the case that $\mathcal{P}(S) \cup \mathcal{P}(T) = \mathcal{P}(S \cup T)$.

(c) (5 points) Is it ever true that $\mathcal{P}(S) \cup \mathcal{P}(T)$ and $\mathcal{P}(S \cup T)$ are equal? If so, describe precisely when this occurs. If not, prove that it is impossible.

Proof. (a) Let $U \in \mathcal{P}(S)$, so $U \subseteq S$. Since $S \subseteq S \cup T$, we have

$$U \subseteq S \subseteq S \cup T$$

and so $U \subseteq S \cup T$, that is, $U \in \mathcal{P}(S \cup T)$. The proof for $U \in \mathcal{P}(T)$ is identical.

(b) Consider the case where $S = \{1\}$ and $T = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(S \cup T)$, but it is not is $\mathcal{P}(S)$ or $\mathcal{P}(T)$.

(c) Yes, there is such a case. We will show that $\mathcal{P}(S) \cup \mathcal{P}(T) = \mathcal{P}(S \cup T)$ if and only if $S \supseteq T$ or $T \subseteq S$.

We'll prove the "if" direction first. Suppose $S \subseteq T$. (The case $T \subseteq S$ is identical, so we can reduce to the case of $S \subseteq T$. This is usually done by saying "We can assume without loss of generality that $S \subseteq T$.") Then $S \cup T = T$ and so

$$\mathcal{P}(S \cup T) = \mathcal{P}(T) \subseteq \mathcal{P}(S) \cup \mathcal{P}(T).$$

Combined with part (a), we have $\mathcal{P}(S) \cup \mathcal{P}(T) = \mathcal{P}(S \cup T)$.

Conversely, suppose that $\mathcal{P}(S \cup T) \subseteq \mathcal{P}(S) \cup \mathcal{P}(T)$. Suppose for the sake of contradiction that $S \not\subseteq T$ and $T \not\subseteq S$. In other words, there exists an $x \in S$ that is not in T and a $y \in T$ that is not in S. Then $\{x, y\} \in \mathcal{P}(S \cup T)$, but $\{x, y\} \notin \mathcal{P}(S)$ and $\{x, y\} \notin \mathcal{P}(S)$, which is a contradiction. □

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3. (20 points) Induction. Prove the following statements by using some form of induction.

(a) (10 points) Show that that $n! < n^{n-1}$ for $n \geq 3$.

(b) (10 points) Show that for any $n \geq 4$, we can obtain exactly n dollars using only \$2 bills and \$5 bills.

Proof. (a) We prove this by induction on $n \geq 3$.

Base case ($n = 3$): We have $6 = 3! < 3^{3-1} = 3^2 = 9$ by inspection.

Inductive step: Assume that our statement holds for n and want to show that it holds for $n + 1$. By the inductive hypothesis, we have $n! < n^{n-1}$. Multiplying both sides by $(n + 1)$, we obtain

$$(n+1)! = n!(n+1) < n^{n-1}(n+1) < (n+1)^{n-1}(n+1) = (n+1)^n,$$

where the last inequality follows from the fact that $n^{n-1} < (n+1)^{n-1}$ for any $n > 1$.

(b) The simplest way to obtain a correct proof of this statement is to use two-step induction (or strong induction) on $n \geq 4$. We'll need more than one base case here, as a result.

Base cases: We note that $4 = 2 + 2$ and $5 = 5$, and so we can obtain 4 and 5 dollars using only \$2 bills and \$5 bills.

Inductive step: Suppose that our statement holds true for $n - 1$ and n, we want to show that it also holds for $n + 1$. By our inductive hypothesis, we can obtain exactly $n - 1$ dollars using only \$2 and \$5 bills. By adding a \$2 bill to this quantity, we obtain $n + 1$ dollars using only \$2 and \$5 bills. $\square$

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4. (20 points) True/False Short Answers about Functions

If True, justify your answer with a proof. If false, give a counterexample (and show why it is indeed a counterexample). (You do not need to salvage false statements.)

(a) Let $n \in \mathbb{N}$ and recall that $[n] = \{1, 2, \ldots, n\}$. If $f : \mathbb{N} \to [n]$ is a function, then f is surjective.

(b) Let $f : X \to Y$ be a function. If A and B are subsets of X and $A \cup B = X$, then $f(A) \cup f(B) = Y$.

(c) If $f : X \to Y$ is injective, then for any subsets $A, B \subset X$, we have $f(A \cap B) = f(A) \cap f(B)$.

(d) (The converse of (c)) Let $f : X \to Y$ be a function with property that $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subset X$. Then f is injective.

Proof. (a) False. Consider the constant map $f(x) = 1$ for all $x \in \mathbb{N}$.

(b) False, because f is not necessarily surjective. (Consider the constant function as above for a concrete counterexample.)

(c) True. Suppose $x \in A \cap B$. Then $x \in A$ and so $f(x) \in f(A)$. Similarly, $x \in B$ and so $f(x) \in f(B)$. Therefore, $f(A \cap B) \subseteq f(A) \cap f(B)$.

For the other inclusion, suppose that $y \in f(A) \cap f(B)$. Then there exists an $x_1 \in A$ such that $f(x_1) = y$ and an $x_2 \in B$ such that $f(x_2) = y$. Since $f(x_1) = y = f(x_2)$ and f is injective, we must have $x_1 = x_2$ and so $x_1 \in A \cap B$. Hence $y = f(x_1) \in f(A \cap B)$.

(d) True. Suppose for the sake of contradiction that f is not injective. We will construct subsets $A, B \subseteq X$ such that $f(A \cap B) \neq f(A) \cap f(B)$. Since f is not injective, there exist points $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$. Let $A = \{x_1\}$ and $B = \{x_2\}$, so $A \cap B = \emptyset$. However, $f(A) \cap f(B) = \{f(x_1)\} \neq \emptyset$, giving us a contradiction. Thus, f must be injective. ☐

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