MATH 3040 Prove It!
Spring 2020

Preliminary Exam #2

INSTRUCTIONS

- You have 60 minutes.
- The exam is closed book, closed notes, no calculators. However, you are allowed a one-page (front and back) "cheat sheet." **If you use such a sheet, submit it with your exam.** You are free to apply any result that we covered in class or on the homeworks, unless the problem explicitly tells you to use a certain approach. You do not need to cite the name and number of such results, just be clear on which result you are using.
- Mark your answers ON THE EXAM ITSELF or on clean sheets of paper (preferably unlined, for legibility). If you are not sure of your answer, you may wish to provide a *brief* explanation so that we can at least know what you are trying to do. For full credit, be sure to justify your steps.
- Write your name on the top of each page with a problem listed.
- Questions are not given in order of difficulty. Make sure to look ahead if stuck on a particular question.

Last Name	
First Name	
Student ID	
All the work on this exam is my own. **(please sign)**	

For staff use only

Q. 1	Q. 2	Q. 3	Q.4	Bonus	Total
/20	/20	/10	/20	/5	/70

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1. (20 points) True/False Short Answers

If True, justify your answer with a proof. If false, give a counterexample (and show why it is indeed a counterexample). (You do not need to salvage false statements.)

(a) The logical negation (if P is a statement, $\neg P$ [i.e. "NOT P"] is its logical negation) of the statement "Every family has its secrets" is "Every family does not have any secrets." (Hint: Translate these into formal logic.)

(b) There is *no* function $f : \mathbf{N} \to [n]$ for any positive integer n that is not surjective.

(c) If A is an infinite set and there is an injection $f : A \to B$, then B has the same cardinality as A.

(d) If there exists an onto function $f : \mathbf{N} \to A$, then A is countable.

Solution. (a) False. For a concrete counterexample, consider the case where there are two families. If one family does not have a secret and the other does, then the property "every family has its secrets" is false, but there still exists a family with a secret. The correct negation is "there exists a family that does not have any secrets."

(b) False. Consider the constant function $f : \mathbf{N} \to [n]$ defined by $f(n) = 1$ for all $n \in \mathbf{N}$.

(c) False. An easy example is the function $f : \mathbf{Z} \to \mathbf{R}$ where $f(x) = x$ for all $x \in \mathbf{Z}$. The set $\mathbf{Z}$ is countably infinite, but $\mathbf{R}$ is uncountably infinite.

(d) True. Since f is surjective, there exists an injective function $g : A \to \mathbf{N}$ where we set a to one of its preimages in $\mathbf{N}$ (there is at least one by surjectivity). Thus, g gives a bijection from A to its image $g(A)$. Since $g(A)$ is a subset of $\mathbf{N}$ and subsets of countable sets are countable, we conclude that A is countable. □

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2. (20 points) Finite Sets and Injectivity/Surjectivity

Let A and B be finite sets.

(a) Suppose that $|A| = |B|$. If $f : A \to B$ is injective, then it is surjective (and thus bijective).

(b) Suppose that $|A| = |B|$. If $f : A \to B$ is surjective, then it is injective (and thus bijective).

Solution. (a) Since f is injective, we have a bijection from A onto its image $f(A) \subseteq B$. Since $|f(A)| = |A| = |B|$, we must have $f(A) = B$, that is, f is surjective.

(b) Here is a different style of argument from part (a) that uses proof by contradiction. Suppose for the sake of contradiction that there f is not injective, so there exists $a \neq a'$ in A such that $f(a) = f(a')$. But then the image of the remaining $|A| - 2$ elements under f cannot contain the remaining $|B| - 1$ elements of B, as

$$|A| - 2 = |B| - 2 < |B| - 1.$$

This contradicts the surjectivity of f. Hence, f must be injective. □

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3. (10 points) Relations

Let $H = \{2^m : m \in \mathbf{Z}\}$. Define a relation on the set $\mathbf{Q}^+$ of positive rational numbers by $a \sim b$ if $a/b \in H$.

(a) Show that $\sim$ is an equivalence relation.

(b) Describe the equivalence class that contains the number 3.

Solution. (a) We need to show that $\sim$ is reflexive, symmetric, and transitive.

Given $a \in \mathbf{Q}^+$, we have $a/a = 1 = 2^0 \in H$, so $\sim$ is reflexive.

Suppose that $a \sim b$, so $a/b = 2^m$ for some $m \in \mathbf{Z}$. Then $b/a = 2^{-m} \in H$ and so $b \sim a$. Thus, $\sim$ is symmetric.

Suppose that $a \sim b$ and $b \sim c$, so $a/b = 2^m$ and $b/c = 2^n$ for some $m, n \in \mathbf{Z}$. Then

$$\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = 2^m \cdot 2^n = 2^{m+n} \in H$$

and so $a \sim c$. Thus, $\sim$ is transitive.

(b) To describe [3], we just need to describe the elements a of $\mathbf{Q}^+$ such that $3/a = 2^m$ where $m \in \mathbf{Z}$. In other words, these are the $a \in \mathbf{Q}^+$ of the form $a = 3 \cdot 2^{-m}$ for all $m \in \mathbf{Z}$; that is

$$[3] = \{3 \cdot 2^n : n \in \mathbf{Z}\}.$$

□

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4. (20 points) Funny Functions

(a) Suppose that a function $f : \mathbf{N} \to \mathbf{N}$ is strictly increasing (i.e. $x < y$ implies that $f(x) < f(y)$) and has the following two properties:

- $f(2) = 2$
- $f(mn) = f(m) \cdot f(n)$ for all $n, m \in \mathbf{N}$.

Show that $f(n) = n$ for all $n \in \mathbf{N}$.

(b) Recall that if A is a set, then $\mathcal{P}(A)$ denotes its power set. Define a function $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ by setting $f(x) = \{y \in \mathbf{R} \mid y^2 < x\}$. Why is this a function? Is f one-to-one? Is it onto?

Solution. (a) We prove this by induction on n.

Base cases: For $n = 1$, we note that since $f(2) = 2$ and f is strictly increasing, we must have $f(1) = 1$. For $n = 2$, our function is defined to have $f(2) = 2$.

Inductive step. Suppose that $f(k) = k$ for all $k \leq n$. We want to show that $f(n+1) = n+1$. Suppose that $n+1$ is not prime, so $n+1 = ab$ for some $a, b < n+1$. Then

$$f(n+1) = f(ab) = f(a)f(b) = ab = n+1.$$

Suppose that $n+1$ is prime. As $n+1 > 2$, we must have $n+1$ be odd. Since f is strictly increasing, to show that $f(n+1) = n+1$, it's enough to show that $f(n) = n$ and $f(n+2) = n+2$. We know $f(n) = n$ by the inductive hypothesis. Since $n+1$ is odd, $n+2$ must be even, so we can write $n+2 = 2m$ for some $m < n$, but then

$$f(n+2) = f(2m) = f(2)f(m) = 2m = n+2$$

by the inductive hypothesis.

(b) This is a function because given any element $x \in \mathbf{R}$, we have $f(x) \in \mathcal{P}(\mathbf{R})$ as $f(x) = \{y \in \mathbf{R} \mid y^2 < x\}$ is a subset of $\mathbf{R}$.

This is not one-to-one. There are no elements $y^2 < x$ for any nonpositive x, and so $f(x) = \emptyset$ for all such $x \leq 0$.

This is not onto. For example, it does not contain the set $\mathbf{R}$ itself, as we note that $f(x) \subset (-x, x)$ and every interval of $(-x, x)$ has finite length (i.e. length $2x$). □

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5. (5 points) Bonus: A Graph Question

This is meant for the people who have already completed the other parts of the exam. It is recommended that you work on those before attempting this problem.

What is the minimum number of vertices n so that *every* graph on n vertices (i.e. with any configuration of edges on it) either contains three vertices $\{x, y, z\}$ that all have edges between each other (i.e. $(x, y), (x, z), (y, z)$ lie in the edge set) or contains three vertices $\{a, b, c\}$ with the property that there is no edge between any of them (i.e. $(a, b), (a, c), (b, c)$ do *not* lie in the edge set)?

Proof. This is now Question 2 on Homework 9. □

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