ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 01/27)

Here are some solutions and commentary to the class assignments from the week of January 27. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

Numbers, Exer 4.3: Prove or disprove: "Where $a, b \in \mathbf{Z}$, the number $a+b$ is even if and only if $a-b$ is even."

We show that the statement is true.

Proof. Let $a, b \in \mathbf{Z}$. We need to prove the result in both directions.

($\Rightarrow$): Suppose that $a+b$ is even, so $a+b = 2 \cdot k$ for some $k \in \mathbf{Z}$. Then

$$a - b = (a+b) - 2b = 2k - 2b = 2(k-b),$$

noting that $k-b$ is an integer, and so $a-b$ is even.

($\Leftarrow$): Suppose that $a-b$ is even, so $a-b = 2 \cdot m$ for some $m \in \mathbf{Z}$. Then

$$a + b = (a-b) + 2b = 2m + 2b = 2(m+b),$$

noting that $m+b$ is an integer, and so $a+b$ is even. $\square$

Numbers, Exer. 6.8: Prove that each nonzero integer has only finitely many divisors.

Proof. Let a be a nonzero integer. Since a is nonzero, $|a|$ is a positive integer. If d divides a, then $|d|$ divides $|a|$ by Exercise 3.2 (applying the first part and the second part of the proposition). Moreover, among the divisors d of a, we have $|d| \in \mathbf{Z}^+$ by Exercise 6.2 and so $|d| \leq |a|$ by Exercise 6.5. Hence, if d is a divisor of a, we must have

$$d \in \{-a, -a+1, \ldots, -1, 1, 2, \ldots, a-1, a\},$$

and this latter set is finite. $\square$

What is an axiom?

Answer. Axioms are statements that are assumed without question or deeper analysis. They are an agreement between the author and reader of which statements are considered "self-evident" and acceptable to both parties, and the mathematical discussion entails what can be logically deduced from the axioms. $\square$

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Remark. A reason why many "discussions" or "debates" in real life lead to arguments and do not resolve in conclusions is because the parties involved do not do this preliminary step of agreeing upon the axioms before the discussion begins. For example, people of different persuasions disagree on whether a proposition like "All human life is equally valuable" should be accepted as an axiom. The gift and the curse of mathematical reasoning (as opposed to physics, philosophy, or other types of inquiry) is that this stage of "agreeing upon axioms" is *not optional*; this allows us to come to conclusive statements, but at the cost of limiting applicability to these ideal situations.

What does it mean that a proposition is deduced from the axioms?

Answer. Essentially, if a proposition is able to deduced from the axioms, it means that there exists a proof of the statement. Namely, if the proposition is of the form "If P, then Q," that there is a sequence of statements $\phi_1, \phi_2, \ldots, \phi_n$ starting with $\phi_1 = P$ and ending with $\phi_n = Q$, where each subsequent statement ("ϕ_{i+1}") is deduced from previous statements ("ϕ_j for $j \leq i$;" or even better, only using "ϕ_i") by applying either the axioms or deductions from propositional logic. Of course, it is often useful to summarize commonly used sequences of statements as a separate proposition and to apply these propositions in an argument, as opposed to always trying to prove statements from first principles. □

What is the difference between a fact, a proposition, and an axiom?

Answer. To understand the distinction deeply is a more philosophical question, but there are also mathematical distinctions between these notions. A fact is a true statement, and so involves an *interpretation* of a sentence, or attaching meaning to it. (In linguistics, these aspects are called "semantics".) A proposition is a sentence with well-defined truth value—it is either true or false in a given context—but strictly speaking, we just think of the proposition as a string of symbols and do not attach a meaning to it. ("Syntax" in linguistics parlance.) For example, "$\exists x, x = -1$" is a proposition that is true in the integers $\mathbf{Z}$ but false in the natural numbers $\mathbf{N}$; determining which set you are working with is what requires an interpretation. An axiom is a proposition that we simply accept as true for our mathematical discussion.

Broadly speaking, the ultimate goal of any mathematical deduction are (1) to find "interesting" true statements and (2) to provide "interesting" proofs of these statements. Determining which facts or arguments are "interesting" is, of course, a matter of context and taste. Sometimes being true or logically sound is enough to be "interesting," but usually there is a rich aesthetic component that allows individuals to exercise their creative freedom and expression. □

What should you say and what should you omit in a proof? How does this differ depending on your audience?

Answer. As you might expect, this is not an easy question to answer, but like all forms of writing, it depends on the audience and the context. The general rule is that you must say enough for both the writer and the reader to see that your argument is correct and properly thought through. In particular, if there is any valid objection from the reader (e.g. "you only addressed the case of even n, what about for odd n?"), it indicates that the proof requires revision.

If there is a common background of accepted facts—in effect, an agreement between the readers and yourself—you are allowed to omit certain routine steps to focus on the essence or novel aspects of the argument. This is most common in specialized settings, say, when you are presenting results to an audience of experts in your research field. $\square$

Remark. For a class like ours, it is much more common for people to omit important details than to have logically sound but lengthy proofs that cover all the bases. So at least for the beginning, we will err on the side of providing too much detail rather than too little, so as to get the in the habit of being careful and remaining vigilant about any potential holes in our arguments.