ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 02/03)

Here are some solutions and commentary to the class assignments from the week of February 3. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

[TAP], Prop. 2.3: $1 \in \mathbf{N}$.

Proof. Suppose for the sake of contradiction that $1 \notin \mathbf{N}$. Since $1 \in \mathbf{Z}$ and $1 \neq 0$, we must have $-1 \in \mathbf{N}$ by Axiom 2.1(iv). By Axiom 2.1(ii), we must then have $(-1)(-1) = 1 \in \mathbf{N}$, which is a contradiction. Hence, we must have $1 \in \mathbf{N}$. □

How to find logically equivalent statements? What statements, aside from "$(A \wedge B) \vee \neg A$" are logically equivalent to "$A \Rightarrow B$"?

Answers. There are infinitely many of these, all of which can verified using truth tables, but some of these include:

- $\neg B \Rightarrow \neg A$
- $\neg A \vee B$
- $\neg(A \wedge \neg B)$

You can also generate further equivalent expressions by taking conjuctions with tautologies (i.e. "always true" statements like "$A \vee \neg A$") or disjunctions with contradictions (i.e. "always false" statements like $A \wedge \neg A$). □

Remark. One underlying principle of this exercise (and the previous one, to some extent) is to illustrate that there are multiple ways to phrase the same mathematical, and that by looking at alternative formulations of the problem, it may suggest a different approach that can lead to a proof that you didn't see when thinking about it from the original perspective.

Modus ponens: If P and $(P \Rightarrow Q)$, then Q.

Proof. Consider the following truth table:

Date: February 10, 2020.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Note that if both P and $(P \Rightarrow Q)$ are true, it must be the case that Q is true. (See the first line of the table.) □

Modus tollens: If $\neg Q$ and $(P \Rightarrow Q)$, then $\neg P$.

Proof. Consider the following truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg Q \wedge (P \Rightarrow Q)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that if both $\neg Q$ and $(P \Rightarrow Q)$ are true, it must be the case that P is false (i.e. $\neg P$ is true). (See the last line of the table.) □

Remark. The use of this logical calculus is one of the principles that makes the foundations of mathematics—and mathematical logic in particular—quite useful and robust as a framework for thought. Instead of having to endlessly prove truth tables for all the mathematical statements that we're trying to prove, just from our bare hands, we can instead reduce down to these basic building blocks of logical deduction and use this to come to conclusions. Of course, humans reasoned long before logic was formalized, but we can now use the framework and machinery of logic and turn it back on our own thinking and sharpen our reasoning!

(Sometimes it's even amazing that such a thing is possible; "modeling our thoughts" is probably a scientific problem for the 21st century and beyond, but even hundreds of years ago with very limited means, we were at least able to construct this mathematical apparatus to be able to get our thinking into written form and create a system for manipulating it that is not only a decent facsimile of our own thinking, but can apply to many other real-world phenomena.)