ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 02/10)

Here are some solutions and commentary to the class assignments from the week of February 17. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

Induction, Exercise 6. Prove the Principle of Mathematical Induction (using Peano's postulates): For each $n \in \mathbf{N}$, let $P(n)$ be a proposition. Suppose that the following two results hold:

(a) The statement $P(1)$ is true.

(b) If $P(n)$ is true, then $P(S(n))$ is true.

Then $P(n)$ is true for all $n \in \mathbf{N}$.

Proof. We want to show that the set

$$B = \{n \mid P(n) \text{ is true.}\} = \{n \mid P(n)\}$$

is precisely the set of natural numbers $\mathbf{N}$. Our assumption (a) says that $1 \in B$ and assumption (b) says that if $n \in B$, then $S(n) \in B$, so by axiom (5) of Peano's postulates, it follows that $B = \mathbf{N}$. □

Comment. The point behind the above exercise, which should be viewed as a quick check of whether you understood the Peano postulates, is that Peano's postulates—beyond giving you a way to characterize the set of natural numbers without making to addition or even the notion of what a *number* is—illustrate some interesting logical interrelations between natural numbers and the logical foundations that make induction possible. Namely, Peano's 5th postulate is essentially equivalent to the principle of mathematical induction, and so can be a viewed as an fundamental property of the natural numbers that actually has little to do with numbers, nor familiar operations on numbers like addition. For those who like thinking about these issues, consider the following question: what other sets (with a distinguished element and a successor function) satisfy the 5th postulate, but are not the natural numbers? (Of course, at least one of the other Peano postulates must necessarily fail for such sets.)

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Induction, Exercise 8. Another chessboard is 2^n squares wide and 2^n squares long. Suppose that one of the squares has been cut out. You have a bunch of L-shaped pieces made up of 3 squares. Prove that you can cover this chessboard with L-shaped pieces with no overlaps for any $n \in \mathbf{N}$.

Proof. We prove this by induction on n.

Base case: ($n = 1$) A 2×2 chessboard with a piece removed consists of a single L-shaped piece. (For example, you could have drawn out the four different possibilities.)

Induction step: Suppose that our statement holds for $2^n \times 2^n$ chessboards, and we want to show that it holds for $2^{n+1} \times 2^{n+1}$ chessboards. Note that a $2^{n+1} \times 2^{n+1}$ chessboard can be divided into four disjoint $2^n \times 2^n$ chessboards that we'll refer to as quadrants. The "missing square" must lie in one quadrant Q, and so Q with the piece removed is a $2^n \times 2^n$ chessboard with a piece removed, so by our inductive hypothesis, we can fill this quadrant with L-shaped pieces. We must now show that we can fill the remainder of the $2^{n+1} \times 2^{n+1}$ chessboard.

Quadrant Q has one corner at the middle of the chessboard. Place a single L-shaped piece just outside this corner of quadrant Q, so that the 4 central squares are now covered. Then the remaining quadrants are now $2^n \times 2^n$ chessboards with one piece removed, which we know we can fill by the induction hypothesis. Hence, we conclude that we can cover a $2^{n+1} \times 2^{n+1}$ chessboard with L-shaped pieces. □

Comment. An important thing to note is how little information is actually needed for the proof. The key part was to discover this inductive or recursive structure of the problem, find a way to exploit it, and somehow turn that into words.

The above proof is an example of a nonconstructive proof. Note that it doesn't, say, give an algorithm for how to tile a given $2^n \times 2^n$ board with L-shaped pieces, but it does show that it is always possible. This is one of the freedoms or strengths of math over something like programming, as it allows you to be concise while remaining rigorous. However, by working through some of the details (for example, going through all cases explicitly), you can turn the argument above into an explicit algorithm. Indeed, there is a close relationship between (formal[1]) proofs and computer programs, called the Curry–Howard correspondence, which underlies many developments in math and computer science and their interactions, especially today.

Going through this kind of "concretization" process often ends up revealing aspects about the problem that were hidden in the original argument and can be interesting in its own right. This kind of "concretization" process

[1] Meaning occurring in a "proof system," which is an explicit set of rules of inference that formalize the logical deductions we use everyday; this kind of topic is a big focus of either a first course in logic or a introduction to proof theory or programming languages, so I'll avoid elaborating further.

is much harder if your nonconstructive proof was a proof by contradiction, which is one reason why direct proofs are preferred when readily available.

Warning: Proofs by contradictions for statements which really don't require them are probably one of the most common ways that somebody in a first class on proofs loses a lot of points, as there are so many ways to mess up—a false step, stating the negation incorrectly, misusing "false" statements that you've proven along the way—especially because the above errors usually do not allow for much partial credit. They are undeniably a powerful tool, but tread carefully and at least try and think about close alternatives like the contrapositive in high stakes situations like exams.

[TAP], Project 5.3. Define the following sets:

$$\begin{aligned} A &:= \{3x : x \in \mathbf{N}\} \\ B &:= \{3x + 21 : x \in \mathbf{N}\} \\ C &:= \{x + 7 : x \in \mathbf{N}\} \\ D &:= \{3x : x \in \mathbf{N} \text{ and } x > 7\} \\ E &:= \{x : x \in \mathbf{N}\} \\ F &:= \{3x - 21 : x \in \mathbf{N}\} \\ G &:= \{x : x \in \mathbf{N} \text{ and } x > 7\} \end{aligned}$$

Determine which of the following set equalities are true.

(i) $D = E$

Solution. False. We have $x = 5 \in \mathbf{N}$ and so $x \in E$, but $x \notin D$ because $x > 7$ is not true. □

(ii) $C = G$

Solution. True. To show this, we must show inclusion in both directions.

("$C \subseteq G$"): Let $x \in C$, so x is of the form $x = y + 7$ for some $y \in \mathbf{N}$. Therefore, $x \in \mathbf{N}$ and $x > 7$ (say, by [TAP], Prop 2.7(i)) and so $x \in G$.

("$G \subseteq C$"): Let $x \in G$, so $x \in \mathbf{N}$ and $x > 7$. We want to show that we can write $x = y + 7$ for some $y \in \mathbf{N}$. We claim that $y = x - 7$ works, but note that subtraction is only defined on the integers $\mathbf{Z}$, not on the natural numbers $\mathbf{N}$ and so we must show that $y = x - 7 \in \mathbf{Z}$ is actually in $\mathbf{N}$. Using [TAP], Prop. 2.13, we see that it is sufficient to show that $y > 0$. Since $x > 7$, we have

$$y = x - 7 > 7 - 7 = 0$$

by Prop. 2.7(i). Hence, we conclude that $x \in C$. □

(iii) $D = B$

Solution. True. Note that $B = \{3x : x \in C\}$ and $D = \{3x : x \in G\}$. (You can show inclusions in both directions for each equality if you want to be careful and super-thorough.) By part (ii), we saw that $C = G$ and so it follows that $D = B$. □

[TAP], Project 5.11.

(i) $A \cap E = B$.

Proof. False. Consider the element $x = 3 \in \mathbf{N}$. It lies in A because $x = 3 \cdot 1$ and in E because it is a natural number, and so is in $A \cap E$. Note that if $y \in B$, then $y > 21$ (e.g. by Prop. 2.17(i)). Since $x > 21$ is not true, x must not be in B. □

(ii) $A \cap C = B$.

Proof. False. Consider the element $x = 9 \in \mathbf{N}$. As $x = 9 = 3 \cdot 3$, we have $x \in A$. As $x = 9 = 2 + 7$, we have $x \in C$. However, 9 is not in B, as if $y \in B$, then $y > 21$. □

(iii) $E \cap F = A$.

Proof. True. We show inclusion in both directions.

("$E \cap F \subset A$"): Let $x \in E \cap F$. Then $x \in \mathbf{N}$ and $x = 3y - 21$, which implies that $y > 7$. Therefore, $x = 3(y - 7)$ with $(y - 7) \in \mathbf{N}$ and so $x \in A$.

("$A \subset E \cap F$"): Let $x \in A$, so $x = 3y$ for some $y \in \mathbf{N}$. Then $x \in \mathbf{N}$, so $x \in E$. Furthermore,

$$x = 3y = 3y + 21 - 21 = 3(y + 7) - 21$$

and $y + 7 \in \mathbf{N}$ and so $x \in F$ as well. □

Sets, Exercise 10. (DeMorgan's law) Let X be a set and let $A, B \subset X$. Then

(1) $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$.

Proof. We can rewrite our sets as follows

$$\begin{aligned} X \backslash (A \cup B) &= \{x \in X : x \notin A \cup B\} \\ &= \{x \in X : x \notin A \text{ and } x \notin B\} \\ &= \{x \in X : x \notin A\} \cap \{x \in X : x \notin B\} \\ &= (X \backslash A) \cap (X \backslash B). \end{aligned}$$

□

(2) $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.

Proof. Like the previous argument, we can rewrite our sets as follows

$$\begin{aligned} X\backslash(A\cap B) &= \{x\in X : x\notin A\cap B\} \\ &= \{x\in X : x\notin A \text{ or } x\notin B\} \\ &= \{x\in X : x\notin A\}\cup\{x\in X : x\notin B\} \\ &= (X\backslash A)\cup(X\backslash B). \end{aligned}$$

□

Comments. The proofs above can also be performed via the standard double inclusion argument to show equality, but using the connection between logic and sets leads to a more streamlined argument.

Functions and Cardinality, Exercise 9. Let A, B, and C be sets and suppose that there is a bijection between A and B, and a bijection between B and C. Then there is a bijection between A and C.

Proof. Let $f : A \to B$ be a bijection from A to B and $g : B \to C$ a bijection from B to C. We want to show that $g \circ f : A \to C$ is a bijection from A to C.

We first show injectivity. Suppose that $(g \circ f)(x) = (g \circ f)(y)$, that is, $g(f(x)) = g(f(y))$. Since g is injective, we must have $f(x) = f(y)$. Since f is injective, it follows that $x = y$. Hence, $g \circ f$ is injective.

It remains to show surjectivity. Let $c \in C$. We want to show that there exists some element $a_c \in A$ such that $(g \circ f)(a_c) = c$. Since g is surjective, there exists a $b \in B$ such that $g(b) = c$. Since f is surjective, there exists an element $a \in A$ such that $f(a) = b$. By setting a_c to be this $a \in A$, we see that $(g \circ f)(a_c) = g(f(a)) = g(b) = c$. Hence, $(g \circ f)$ is surjective. □