ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 03/09)

Here are some solutions and commentary to the class assignments from the week of March 9. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

Functions and Cardinality, Theorem 10. (Pigeonhole Principle) Let $n, m \in \mathbf{N}$ such that $n < m$. There does not exist an injective function $f : [m] \to [n]$.

Proof. Suppose for the sake of contradiction that there exists an injective function $f : [m] \to [n]$. Then if $x, y \in [m]$ and $x \neq y$, then $f(x) \neq f(y)$. (This is just the contrapositive of the usual definition of injection.) Therefore, there must exist at least m distinct elements in $[n]$, which contradicts our assumption that $n < m$. Thus, there is no injection $f : [m] \to [n]$. □

Infinities, Theorem 3. Let A be a set. The following statements are equivalent:

(i) The set A is infinite.
(ii) There exists an injective function $f : \mathbf{N} \to A$.
(iii) There exists a one-to-one correspondence between A and a proper subset of A.

Proof. $(i) \Rightarrow (ii)$: We want to inductively construct an injective function $f : \mathbf{N} \to A$. Pick $x_1 \in A$. Define $f(1) = x_1$. Pick an $x_2 \in A \backslash \{x_1\}$ and define $f(2) = x_2$. We want to show that we can continue this process for all $n \in \mathbf{N}$. To do so, it suffices to show that

$$B_n = A \backslash \{x_1, x_2, \ldots, x_n\}$$

is nonempty for all $n \in \mathbf{N}$, no matter how we choose the x_i's.

Suppose for the sake of contradiction that B_n is empty for some n. Then there exists an an integer $m \leq n$ such that $[m]$ is in bijection with A. But this would mean that A is finite, which contradicts our assumption that A is infinite.

$(ii) \Rightarrow (iii)$: Let $f : \mathbf{N} \to A$ be an injective function. We need to construct a bijection from A to a proper subset of A. Consider the function $g : A \to A \backslash \{f(1)\}$ defined by

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n) \text{ for some } n \in \mathbf{N}, \\ x, & \text{otherwise.} \end{cases}$$

We want to show that g is a bijection. (If you want to be really careful, you should also show that g is well-defined; that is, for all x, the element $g(x)$ lies in $A\backslash\{f(1)\}$ and if uniquely determined by x, but we'll omit this check here as it should be immediate that this is the case for g.)

For injectivity, suppose that $g(x) = g(y)$. We have two cases depending on where $g(x)$ lies in $A\backslash\{f(1)\}$.

Case 1. If $g(x) \in f(\mathbf{N}) \cap A\backslash\{f(1)\} = f(\mathbf{N}\backslash\{1\}) \cap A$, then we know that $g(x) = f(m)$ for some $m \neq 1$ and so $g(x) = f(n+1)$ for some $n \in \mathbf{N}$. By the definition of g, this implies that $x = f(n) = y$.

Case 2. If $g(x) \notin f(\mathbf{N})A\backslash\{f(1)\}$, then $g(x) = x$. Therefore, $x = g(x) = g(y) = y$.

Hence, g is injective.

For surjectivity, let $y \in A\backslash\{f(1)\}$. We have two cases here as well.

Case 1. If $y \in f(\mathbf{N}) \cap A\backslash\{f(1)\}$, then $y = f(m)$ for some $m > 1$ and so $y = f(n+1)$ for some $n \in \mathbf{N}$. Then $g(f(n)) = f(n+1) = y$.

Case 2. If $y \notin f(\mathbf{N}) \cap A\backslash\{f(1)\}$, then $g(y) = y$.

Hence, g is surjective.

(We can actually do this for any infinite proper subset $E \subset \mathbf{N}$, by replacing $A\backslash\{f(1)\}$ with $\mathbf{N}\backslash\{E^c\}$ and modifying g appropriately with a successor function defined on E.)

(iii) $\Rightarrow$ *(i)*: We prove the contrapositive. Suppose that A is finite. If $S \subset A$ is a proper subset of A, we have $|S| < |A|$ and so there cannot exist a bijection between S and A. □

Infinities, Example 5. The following sets are countable:

(1) The set of all odd numbers.

Proof. We have a bijection given by mapping the natural numbers $1, 2, 3, 4, \ldots$ to $1, -1, 3, -3, 5, -5, \ldots$ and so on. □

(2) Prime numbers.

Proof. We have a bijection from $\mathbf{N}$ to the prime numbers by mapping n to the nth (smallest) prime number. (*Note that we didn't need to give a formula or procedure to define a function; this is one of the strengths and flexibility of mathematics.*)

If you want to be careful and check that the function is well-defined, you need to make sure that the procedure you give to find the nth prime number works. For example, one explicit way to find the numbers is to look at $\mathbf{N}$ and (after skipping one) inductively pick the next available number (which is guaranteed to be prime) then remove all its multiples from $\mathbf{N}$. (This procedure—one of the first known algorithms—is usually referred to as the *sieve of Eratosthenes* and dates back to at least the 3rd century BCE.) □

(3) The integers.

Proof. We have a bijection $f : \mathbf{N} \to \mathbf{Z}$ given by $0, 1, -1, 2, -2, 3, -3, \ldots$, etc. □

(4) All students in this class.

Proof. This is finite and so is countable. □

(5) All possible letter combinations using any finite alphabet.

Proof. Suppose that we have letters $a_1, \ldots, a_n$ in our alphabet. We can put these in lexicographic order (i.e. a_1 comes before a_2, which comes before a_3, etc). We construct a function from $\mathbf{N}$ to the set of all words formed by letters in our alphabet as follows by first going through all the one-letter words in lexicographic order, then all the two-letter words in lexicographic order, then all the three letter words... □

Remark. The trick to showing that all of these things are countable is to be able to say what "the next" element is (i.e. a successor function). This is what cannot be done explicitly for uncountable sets. For example, what's "the next biggest" real number after 0?

Infinities, Proposition 7. Every subset of a countable subset is countable.

Proof. Let A be a countable set. If A is finite, then for any subset $S \subset A$, we have $|S| \leq |A|$ by Exercise 12 from the *Functions and Cardinality* sheet. It remains to address the case where A is countably infinite.

Since A is countably infinite, there is a bijection between A and $\mathbf{N}$. Therefore, it suffices to prove that every subset of $\mathbf{N}$ is countable. If $S \subset \mathbf{N}$ is finite, then it is countable, so it only remains to address the case where $S \subset \mathbf{N}$ is infinite. We want to show that S must be countably infinite. We will do so by constructing a bijection from $\mathbf{N}$ to S.

Recall that for any nonempty subset $B \subset \mathbf{N}$, there exists a smallest element $\min B \in \mathbf{N}$ (as $\mathbf{N}$ has the natural ordering by size). Define a function $f : \mathbf{N} \to S$ where

$$f(n) = \min S \backslash \{f(1), f(2), \ldots, f(n-1)\}$$

We want to show that this is a bijection. It immediate that this is a injection, since $f(x) \neq f(y)$ for any $x \neq y \in \mathbf{N}$.

It remains to show that it is surjective. Let $y \in S \subset \mathbf{N}$. There are at most $y - 1$ elements less than y in $\mathbf{N}$, so we must have $y \in \{f(1), f(2), \ldots, f(y)\}$. If not, then either y does not lie in S (a contradiction) or $y > y$ as a natural number (also a contradiction).

Hence, we have a bijection $f : \mathbf{N} \to S$ and so S must be countably infinite. □

Remark. In the proof above, what was the key point where the hypotheses that $\mathbf{N}$ was countable was used? It was when we deduced that y must lie in $\{f(1), f(2), \ldots, f(y)\}$, which relies on the ordering on $\mathbf{N}$ by size.