ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 04/13)

Here are some solutions and commentary to the class assignments from the weeks of April 13. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

Relations, Theorem 12. (Fundamental Theorem of Equivalence Relations.) Let A be a nonempty set. If R is an equivalence relation on A, then the equivalence classes (with respect to R) form a partition of A. Conversely, if S is a partition of A and we define a relation R on A such that $a \sim b$ if and only if a and b are in the same set in the partition, then R is an equivalence relation on A.

Proof. Suppose that R is an equivalence relation on A; we need to show that $S = \{[a] : a \in A\}$ is a partition of A.

Since R is reflexive, we must have $a \sim a$ and so $a \in [a]$, implying that $\emptyset \notin S$.

Suppose that $x, y \in S$, so we can write $x = [a]$ and $y = [b]$ for some $a, b \in A$. We have two cases to address, when a is related to b and when a is not related to b. If $a \sim b$, then $[a] = [b]$. Now suppose that $a \not\sim b$. We need to show that $[a] \cap [b] = \emptyset$. Suppose that this were nonempty, and let $c \in [a] \cap [b]$, so $a \sim c$ and $c \sim b$. But then $a \sim b$ by transitivity of R, which contradicts the our assumption that $a \not\sim b$. Thus, we must have $[a] \cap [b] = \emptyset$ when $a \not\sim b$.

Finally, we need to show that $\bigcup_{s \in S} s = A$. As S contains subsets of A, we have $\bigcup_{s \in S} s \subseteq A$. For the opposite direction, we note that if $a \in A$, then $a \in [a] \in S$.

Now, suppose that S is a partition of A with the relation $a \sim b$ if and only if a, b lie in the same $s \in S$. We need to check that it's an equivalence relation.

For reflexivity, we note that if $a \in A$, then a is in some $s \in S$ by property (3) of being a partition, and so $a \sim a$.

For symmetry, suppose that $a \sim b$. Then a, b lie in some $s \in S$, and so $b \sim a$.

For transitivity, suppose that $a \sim b$ and $b \sim c$. Then a, b lie in some $s \in S$ and b, c lie in soe $t \in S$. Since S is a partition and $b \in s \cap t$, we must have $s = t$ and so $a \sim c$. □

Remark. Not every mathematical subject has a "Fundamental Theorem" (or set of "Fundamental Theorems") but the goal of any such subject is to be able to get to such a point. Most math subjects that are currently taught at the undergraduate level are in this mature form. What are the "Fundamental Theorems" from the math classes that you have taken thus far? What should be the "Fundamental Theorems" for the other topics that we have tackled thus far in this course?

Graphs, Proposition 8. Let $G = (V, E, F)$ be a connected planar graph. Assume that $|V| \geq 3$. then $|E| \geq \frac{3}{2}|F|$.

Proof. We address two cases: one where there is only one face (i.e. G is a tree), and one where we have more than one face.

Case 1: Tree. Suppose that G is a tree. Then $|F| = 1$ and $|E| = |V| - 1$ by Theorem 6. Since $|V| \geq 3$, we have

$$|E| = |V| - 1 \geq 3 - 1 = 2 = 2|F|.$$

Case 2: Non-tree. Suppose that G has more than one face (note that this also requires that $|V| \geq 3$). Since each face is bounded by at least three edges, we can write

$$|F| = \sum_{n=3}^{\infty} F_n.$$

where F_n denotes the number of faces bounded by n edges. Since each edge contributes to the boundary of exactly two faces, we have

$$\sum_{n=3}^{\infty} nF_n = 2|E|.$$

Therefore,

$$2|E| = \sum_{n=3}^{\infty} nF_3 \geq \sum_{n=3}^{\infty} 3F_n = 3|F|,$$

which implies that $|E| \geq \frac{3}{2}|F|$. □

Remark. It is much easier to delude yourself into thinking that your incorrect proof is correct if you don't split your work into cases like this. Many arguments in what we consider "discrete mathematics" (like graphs) as opposed to "continuous mathematics" (like calculus), require this kind of casework, so you should be very suspicious of your arguments if the result you're trying to prove is quite general and you do not have cases in your proof.