ANSWERS TO CLASS ASSIGNMENTS (WEEK OF 05/05 AND 05/12)

Here are some solutions and commentary to the class assignments from the weeks of May 5 and 12. These are not the only way to do it, and they are not even "model solutions"—your solution may be better in a number of ways, and these might not necessary be "full credit" answers (especially with respect to the meta-mathematical questions)—but they give an idea of what we'd consider an acceptable answer to certain questions.

Surreal Numbers, Question 7. Show that $\frac{1}{2} + \frac{1}{2} = 1$.

Proof. This is a good exercise to test your logic and induction skills, as it involves carefully stepping through chains of inequalities.

Recall that $\frac{1}{2} = \{0|1\}$ (created on day 2). By adding these together, we obtain

$$\frac{1}{2} + \frac{1}{2} = \left\{ \frac{1}{2} \middle| 1 + \frac{1}{2} \right\},$$

where for the left, we are using that $x + 0 = x$ for all surreal numbers x. (Prove it if you haven't done so!) Write $\alpha = 1 + \frac{1}{2}$. We know in our mind that it should be $\frac{3}{2}$, but we haven't proven this yet. One can show that $\alpha = \frac{3}{2} = \{1|2\}$ and use the "simplicity theorem" to show that $\frac{1}{2} < 1 < \frac{3}{2}$, giving us the result, but here is a more elementary and direct approach that doesn't rely on these general results.

Let's first show that $\frac{1}{2} + \frac{1}{2} \geq 1$. This holds unless we have $\alpha \leq 1$ or $\frac{1}{2} + \frac{1}{2} \leq 0$, so we need to check these:

- Is $1 \geq \alpha$? Yes, unless $1 \leq \alpha^L$ for some α^L (because there is no 1^R, the "higher" condition for $\geq$ is vacuous). But one α^L is $1 + 0 = 1$. Thus, we conclude that $1 \not\geq \alpha$.
- If $0 \geq \frac{1}{2} + \frac{1}{2}$? Yes, unless $0 \leq (\frac{1}{2} + \frac{1}{2})^L$ for some $(\frac{1}{2} + \frac{1}{2})^L$ (because there is no 0^R). But since $0 \leq \frac{1}{2} + 0 = \frac{1}{2}$ and $\frac{1}{2}$ is a $(\frac{1}{2} + \frac{1}{2})^L$, we have $0 \not\geq \frac{1}{2} + \frac{1}{2}$.

Thus, we have $\frac{1}{2} + \frac{1}{2} \geq 1$.

Checking that $\frac{1}{2} + \frac{1}{2} \leq 1$ is completely analogous. $\square$

Remark. Note that there are merits to both an elementary approach and in proving a general result and then applying it. The former is certainly more concrete and is probably necessary to go through to find the patterns to deduce the general result, but going through the details can sometimes confuse as much as illuminate. (The main goal of calculations and working through examples is to provoke questions and lead to understanding.) The latter is how we should mark the progress that we've made, by proving a claim that encapsulates the knowledge we've gathered and the key properties we've identified into a definitive result that we can apply and understand even if we step away from being immersed in the problem and so forget about our more explicit investigations. However, the general results can be rather opaque and "brittle," and hard to extend to new situations if you have not done the legwork of understanding examples or what really goes into the proof.

For example, there's a wide gulf between knowing each of how to take derivatives of polynomials, how to apply the fundamental theorem of calculus, and understanding the (relatively complicated) definition of a derivative. If you only know the first, it's hard to see how to take the derivative of a general differentiable function. If you only know the second, it's hard to see how this applies to a new situation you haven't seen in class (e.g. non-differentiable functions, an application outside of the ones that are told to you in physics or economics). But if you really understand the third, you really are grasping something deep about calculus and it is easy to "get" the first two. However, how can you test if somebody really understands what a derivative is? You really can't—at least, not in a constrained weekly homework or testing environment—and so you have to settle for testing for the former two, which are the bare minimum needed for classes that build upon calculus—and just hope that those who really care will go the extra mile or perhaps learn it eventually if they need it.

For a lot of undergraduate math courses—including much of the content for this class—one can easily get away with only tackling the formal parts, as those are primarily what's tested and what tends to be uniform for everybody in the class. Indeed, being comfortable with such abstractions is one of the key goals of math courses. However, do not be misled into thinking that what's "testable" is what's most important. Just memorizing the proofs of formal results (or worse, just the "big theorems") without taking the time to work out examples, make lots of mistakes, and internalize the concepts leads to a very shallow understanding that is revealed when you actually have to apply your knowledge or even actually compute something.

...and Beyond, Question 4. What is the nimber associated with the bridge standoff game of length 8? Of length 21? For arbitrary n?

Solution. Probably the best way to approach this problem was to just start with the smallest examples, identify their nimbers, and then try to generalize. From the first of n, we can work things out recursively:

$$\begin{aligned}
B(3,0) &= \{\} = *0 \\
B(3,1) &= \{*0\} = *1 \\
B(3,2) &= \{*0, *1\} = *2 \\
B(3,3) &= \{*0, *1, *2\} = *3 \\
B(3,4) &= \{*1, *2, *3\} = *0
\end{aligned}$$

and we from this we can see that the nimbers are periodic with period 4, so

$$B(3,n) = *(n \bmod 4)$$

where "n mod 4" denotes the unique integer $k \in \{0,1,2,3\}$ such that $4 \mid (n-k)$.

There are many ways to prove this. For one, use the observation that

$$B(3,k+1) = \{B(3,k-2), B(3,k-1), B(3,k)\}$$

so we have $B(3,k+1) \in \{*0, *1, *2, *3\}$ and then you prove that $B(3,k+1) = *(B(3,k) + 1 \bmod 4)$ just by induction. $\square$

Remark. An interesting fact about playing multiple instances of the bridge standoff game simultaneously—these are variants of the game Nim, whence the name "nimbers"—is that if you don't know the strategy, it is difficult to tell who has the stronger position. For example, suppose that we are playing three instances $B(3,5)$, $B(3,6)$, $B(3,2)$. We have

$$B(3,6) + B(3,5) + B(3,2) = *2 + *1 + *2 = *1$$

as $*n + *n = 0$ for all n, and so the first player has a winning strategy. To find a winning move, the first player must find a way to move to a position with value $*0$, such as jumping one step in the second game (moving from $B(3,5)$ to $B(3,4)$). From here, one can show that the first player has a winning response to any move by the other play (just counter by forcing a move to a position with value $*0$).